(1) (*8 marks*) Consider the function

$$f(x) = ax^2 + \frac{b}{x}$$

where a and b are constants. Find $f'(x)$ using the definition of the derivative (Newton's quotient).

(2) (*8 marks*) Determine the values of a and b so that $\lim_{x\to\infty} \sqrt{x^2 - 5x + 1} - ax + b = 0$

(3) (*12 marks*) Evaluate the following limits and simplify your answers. You may use L'Hopital's rule where appropriate.

(a) $\lim_{x\to 0^+} (\sin x)^{\tan x}$

(b) $\lim_{t\to 1} \dfrac{nt^{n+1} - (n+1)t^n + 1}{(t-1)^2}$ for $n \geq 2$

(c) $\lim_{x\to 0^+} \sqrt[3]{x^2 + x}\, \sin\left(\dfrac{1}{x^2}\right)$

(4) (*8 marks*) A container in the shape of a right circular cone with vertex angle a right angle is partially filled with water.

(a) Suppose water is added at the rate of $3\, cm^3/s$. How fast is the water level rising when the height $h = 2\, cm$?
(b) Suppose instead no water is added, but water is being lost by evaporation. Show the level falls at a constant rate.

(5) (*8 marks*) Consider the function

$$f(x) = \begin{cases} \dfrac{\sin^2 x}{x} & ; \quad x < 0 \\ \ln(1 + a\tan x + b\cos x) & ; \quad x \geq 0 \end{cases}$$

Determine the values of the constants a and b that will make the function $f(x)$ continuous and differentiable on the interval $(-1, 1)$.

(6) (*8 marks*) The equation

$$x^5 + x^2 y + y^3 = 4y + 3$$

defines y implicitly as a function of x near the point $(x, y) = (1, 2)$.

(a) Determine the value of y' at $(x, y) = (1, 2)$
(b) Determine the value of y'' at $(x, y) = (1, 2)$
(c) Estimate the value of y when $x = 0.97$ by using linear approximations/differentials and the information from part (a).

(7) (*8 marks*) Let $f(x) = |x^2 - 5x - 6| + 2x^2 + 17x$. Find the absolute maximum and absolute minimum of $f(x)$ for $-6 \leq x \leq 3$ and the locations where they occur.

(8) (*8 marks*) A tall, open pot has the shape of a cylinder, with a circular base of radius a inches. A cabbage with radius r inches, where $0 < r < a$, is placed in the pot, and the pot is filled with just enough water to cover the cabbage completely. What is the radius of the cabbage which requires the greatest amount of water to accomplish this?

(9) (*12 marks*) Consider the function

$$f(x) = \frac{ax^2}{x^2 + b^2},$$

where a is a positive constant and b is a constant.

(a) Determine the domain of $f(x)$
(b) Determine its asymptotes.
(c) Determine the intervals of increasing/decreasing.
(d) Determine the intervals of concavity.
(e) Classify all extremas for this function (local or absolute)
(f) Identify all inflection points.

(10) (*8 marks*) Assume that $f(x)$ is continuous on $[a, b]$ and differentiable on (a, b). Also assume that $f(a)$ and $f(b)$ have opposite signs and $f'(x) \neq 0$ between a and b. Show that $f(x) = 0$ exactly once between a and b.

(11) (*12 marks*) In each part below, use the method of your choice (that works) to find the derivative. You do not have to simplify your solution.

(a) Determine $f'(x)$ if $f(x) = \sqrt{1 - \sqrt{1 - \sqrt{1 - x^4}}}$

(b) Determine $g'(x)$ if $g(x) = \dfrac{e^x \cdot 5^{x^2} \cdot 11^x}{4^{x+8} \cdot 3^{x^3}}$

(c) Determine $\dfrac{d}{dx}(f^{-1})(2)$ if $f(x) = x + x^3$