DIFFERENTIAL EQUATIONS, H20, TEST 2

(1) (10 marks) i) Using the method of undetermined coefficients prove (show all steps) that the solution of the initial-value problem

$$x'' + \omega_0^2 x = F_0 \cos \omega t, \quad x(0) = 0, \; x'(0) = 0$$

is

$$x(t) = \frac{2F_0}{\omega^2 - \omega_0^2} \sin \frac{1}{2}(\omega - \omega_0)t \sin \frac{1}{2}(\omega + \omega_0)t$$

ii) Draw the graph of the solution $x(t)$ for the case when $\omega - \omega_0$ is relatively small compared to $\omega + \omega_0$.

iii) Take the limit

$$\lim_{\omega \to \omega_0} x(t)$$

iv) Draw the graph of the resulting limiting solution.

(2) (8 marks) Determine the steady-state charge and the steady-state current in an LRC-cicuit when $L = 1H$, $R = 2\Omega$, $C = 0.25F$, and the driving voltage is $V(t) = 50\cos t$ volts.

(3) (8 marks) Consider the equation for free mechanical vibration,

$$my'' + \gamma y' + ky = 0,$$

and assume that the motion is critically damped. Let $y(0) = y_0 \neq 0$ and $y'(0) = v_0$. Prove that the mass will pass through its equilibrium at exactly one positive time iff

$$\frac{2my_0}{2mv_0 + \gamma y_0} < 0$$

(4) (8 marks) Solve the initial value problem $\mathbf{x}'(t) = A\mathbf{x}(t)$,

$$A = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix}, \qquad \mathbf{x}(0) = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

Draw the phase portrait of this linear system of DE's emphasizing the particular trajectory selected by the initial conditions.

(5) (8 marks) Determine the general solution of $\mathbf{x}'(t) = A\mathbf{x}(t)$,

$$A = \begin{pmatrix} 1 & -2 & 2 \\ -2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}.$$

(6) (10 marks) Determine the general solution of $\mathbf{x}'(t) = A\mathbf{x}(t)$,

$$A = \begin{pmatrix} 5 & -4 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 5 \end{pmatrix}.$$

(7) (8 marks) Solve the initial value problem $\mathbf{x}'(t) = A\mathbf{x}(t)$,

$$A = \begin{pmatrix} 2 & 8 \\ -1 & -2 \end{pmatrix}, \qquad \mathbf{x}(0) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

Draw the phase portrait of this linear system of DE's emphasizing the particular trajectory selected by the initial conditions.

(8) (10 marks) For the system of DE's $\mathbf{x}'(t) = A\mathbf{x}(t)$,

$$A = \begin{pmatrix} 4 & \alpha \\ 8 & -6 \end{pmatrix}$$

i) Find the critical values of α where the qualitative nature of the phase portrait for the system changes.
ii) Sketch a phase portrait for each critical value of α as well as for one α for each interval between critical values.

(9) (10 marks) Consider the system of linear DE's

$$\mathbf{x}' = A\mathbf{x}, \quad A = \begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix}.$$

i) Compute the fundamental matrix $\Phi(t) = e^{At}$.
ii) Solve this system with the initial conditions

$$\mathbf{x}(0) = \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

(10) (10 marks) Determine the general solution of the nonhomogeneous system of DE's

$$\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} t \\ t+1 \end{pmatrix}$$

by using diagonalization.

(11) (10 marks) Determine the general solution of the nonhomogeneous system of DE's

$$\mathbf{x}' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} \sec t \\ 0 \end{pmatrix}$$

by using the method of variation of parameters.