201-NYA-05 Calculus I

Final Examination

23 December 2021

Name:

This examination consists of 13 questions for a total of 100 points.
You will have **180 minutes** to complete the test.

Instructions:

- Write your answers directly on the questionnaire.
- Show all work. Your solutions will be scored on the correctness and completeness of your methods and use of proper notation as well as your answers.
- Write neatly and legibly. Draw large, clear diagrams where required.
- Only non-graphing, non-programmable calculators are permitted. All other electronic devices including (but not restricted to) smart phones, smart watches, recording and/or playback devices in any form are expressly prohibited and must be placed out of reach during the entire exam. Possession of any of these items will be considered cheating.
- Give exact answers. $\sqrt{2}$ is exact, 1.41 is an approximation of $\sqrt{2}$.
- Simplify all answers where possible.

Note:

- Start by reading over the entire test.
- Start with a question you find easy.

Good Luck!

Cheating and plagiarism are serious academic offenses. Anyone caught cheating, or aiding in the act of cheating, will immediately be given a mark of zero for this test, and a note will be placed in his or her file.

Marks	
1	/ 15
2	/ 5
3	/ 6
4	/ 12
5	/ 5
6	/ 6
7	/ 6
8	/ 5
9	/ 6
10	/ 6
11	/ 5
12	/ 6
13	/ 17
Total:	/100

1. Evaluate the following limits. Indicate any infinite limit with $+\infty$ or $-\infty$ as appropriate, and identify any limits that do not exist. **/15**

(a) $\displaystyle\lim_{x\to 2^-} \frac{|x^2 - x - 2|}{|x^2 - 5x + 6|}$

(b) $\displaystyle\lim_{x\to 3} \frac{|x^2 - x - 2|}{|x^2 - 5x + 6|}$

(c) $\lim_{x \to 4} \left(\frac{1}{\sqrt{x} - 2} - \frac{4}{x - 4} \right)$

(d) $\lim_{x \to \infty} \frac{1 + \sin 2x}{1 + x}$

(e) $\lim_{x \to 0} (\sec x)^{2/x^2}$

/5

2. Determine the values of the constants a and b for which the function

$$f(x) = \begin{cases} ax - b & \text{if} \quad x \leq -1 \\ 2x^2 + 3ax + b & \text{if} \quad -1 < x \leq 1 \\ 4 & \text{if} \quad x > 1 \end{cases}$$

is continuous for all values of x.

3. Let $f(x) = \sqrt{2 - x^2}$

/**6**

(a) Calculate $f'(1)$ using the **limit definition** (Newton's quotient).

(b) Verify your answer to (a) using differentiation rules.

(c) Find an equation for the tangent line to the graph $y = f(x)$ at the point where $x = 1$.

/12

4. Calculate the derivative $y' = \dfrac{dy}{dx}$. **Do not** simplify your answers.

(a) $y = \dfrac{2}{x} + 3\ln(2x-1) + 2^{-4x+1}$

(b) $y = \dfrac{1 - x\arcsin\left(x^2\right)}{2 - x^3}$

(c) $y = (\cot x)^{\sec 2x}$

(d) $y = \dfrac{3^x \left(x^2 - 1\right)^6}{\sin(6x)\sqrt[5]{5 - 2x}}$ (You must use logarithmic differentiation.)

/5

5. Determine an equation for the tangent line to the curve described by the equation

$$2x^4 + y^3 = 5xy$$

at the point $(1, 2)$.

6. (a) State the Extreme Value Theorem.

/**6**

(b) Explain why the function $f(x) = 3x^{2/3} + x$ must attain and absolute maximum and and absolute minimum value on the interval $[-10, 1]$.

(c) Find the absolute maximum and and absolute minimum value of the function $f(x) = 3x^{2/3} + x$ on the interval $[-10, 1]$.

/6

7. Argue that the equation $2021x = 12 + 23\cos x$ has *precisely* one solution.

8. Two straight roads intersect at right angles. At 10am a car passes through the intersection headed due East at 90 km/h. At 10:30am a truck heading due North at 80 km/h passes through the same intersection. Assume that the two vehicles maintain their speeds and directions. **/5**

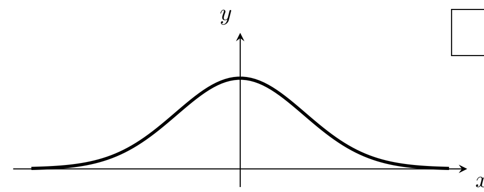
   (a) What is the distance between the two vehicles at 11:30pm?

   (b) At what rate are they separating at 11:30pm?

/6

9. A factory is on one bank of a straight river that is 2 km wide. On the opposite bank but 4.5 km downstream is a power station from which the factory draws its electricity. It costs three times as much per meter to lay an underwater cable as to lay cable above ground. What path should a cable connecting the power station to the factory take so that the cost of laying the cable is minimized?

10. Determine the point(s) on the graph of $f(x) = e^{-x^2}$ (shown on the right) that are closest to the origin.



/6

/5

11. (a) Estimate $\sqrt{25.2}$ using the linear approximation $L(x)$ of $f(x) = \sqrt{x}$ at $a = 25$.

    (b) Is the estimate in part (a) larger or smaller than the actual value of $\sqrt{25.2}$? Which property of f is responsible? (*Hint: Look at the graph, and/or calculate* $f''(x)$.)

12. (a) Sketch the graph of $y = \arctan x$. State its domain and range. **/6**

Domain:

Range:

(b) State and prove a formula for the derivative of $y = \arctan x$.
(*Hint: You may make use of the formula for the derivative of* $x = \tan y$.)

/17

13. Given $f(x) = \left(x^3 - 1\right)^{2/3}$ with $f'(x) = \dfrac{2x^2}{(x^3-1)^{1/3}}$ and $f''(x) = \dfrac{2x^4 - 4x}{(x^3-1)^{4/3}}$

(a) State the domain of f and find the x and y intercepts, if any.

(b) Is f even, odd, or neither? Justify.

(c) Does the graph of f have any horizontal or vertical asymptotes? Justify briefly.

(d) Where is f differentiable? Justify briefly.

Recall $f(x) = \left(x^3 - 1\right)^{2/3}$ with $f'(x) = \dfrac{2x^2}{\left(x^3 - 1\right)^{1/3}}$ and $f''(x) = \dfrac{2x^4 - 4x}{(x^3 - 1)^{4/3}}$

(e) Determine the intervals where f is increasing and decreasing, and find the (x, y) coordinates of any local extreme values.

(f) Determine the intervals of concavity and find any inflection points. *(Hint: Use test values to determine signs. For the inflection point(s) only: You may give an approximate answer to two decimals.)*

(g) Summarize your findings of the sign-variation of f' and f''.

(h) Sketch the graph of f. Choose an appropriate scale and make full use of the given space. Clearly identify and label all features you found in parts (a)–(f).

(Space for calculations)

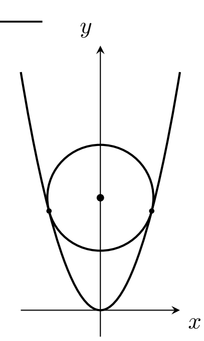
(Space for calculations)

(Space for calculations)

(Space for calculations)

Please only attempt to solve the following **bonus question** if you have finished the test well before the time limit and if you have carefully proofread all your answers. **No partial credit** for this question.

A circle of radius 1 with centre on the y-axis is inscribed in the parabola $y = 2x^2$. Find the points of contact.



End of Examination

Have a great winter holiday!