DIFFERENTIAL EQUATIONS, H22, TEST 2

Name: ________________________ Student number______________________

(1) (3.5 marks) The coefficient matrix of the following system of differential equations depends on a parameter α.
a) Determine the eigenvalues in terms of α.
b) Find the critical values of α where the qualitative nature of the phase portrait for the system changes.
c) Draw qualitative phase portraits for this system for values of α taken in the intervals outside of the critical points.

$$\mathbf{x}' = \begin{pmatrix} 2 & -5 \\ \alpha & -2 \end{pmatrix} \mathbf{x}.$$

(2) (4.5 marks) The motion of a mechanical system driven by a periodic external force can be modelled as a initial value problem

$$u'' + \omega_0^2 u = F \cos \omega t, \quad u(0) = 0,\ u'(0) = 0,$$

where F, ω_0 and ω are constants and $\omega \neq \omega_0$.

a) Solve the initial value problem. The result should be a function $u(t)$ with the constants F, ω_0, ω as parameters in the function.

b) Now let $F = 1$, $\omega_0 = 1$ and $\omega = 1.1$. Substitute these values in your solution $u(t)$. Sketch the graph of $u(t)$ clearly indicating the maximum value(s) on the graph and the period if $u(t)$ is periodic.

(3) (4 marks) Solve the initial value problem $\mathbf{x}'(t) = A\mathbf{x}(t)$,

$$A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \qquad \mathbf{x}(0) = \begin{pmatrix} -1 \\ -2 \end{pmatrix}.$$

Draw the phase portrait of this linear system of differential equations emphasizing the particular trajectory selected by the initial conditions.

(4) (4 marks) a) Find the solution of the initial value problem

$$\mathbf{x}' = \begin{pmatrix} -2.5 & 1.5 \\ -1.5 & 0.5 \end{pmatrix} \mathbf{x}, \qquad \mathbf{x}(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

b) Draw a phase portrait of this system of DE's emphasizing the trajectory selected by the initial conditions.

(5) (4 marks) Determine the matrix exponential e^{At} for the matrix A:

$$A = \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix}.$$

Next use this result to solve the initial value problem

$$\begin{aligned} x_1' &= 5x_1 - x_2 \\ x_2' &= 3x_1 + x_2, \end{aligned} \qquad x_1(0) = -2,\ x_2(0) = 4.$$

(6) (4 marks) Determine the solution of the initial value problem

$$\mathbf{x}' = \begin{pmatrix} 2 & -3 \\ 1 & -2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} e^{2t} \\ 1 \end{pmatrix}, \qquad \mathbf{x}(0) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}.$$

(7) (4 marks) Use Laplace transform to solve the initial value problem

$$y'' - 2y' + 5y = -8e^{-t}, \qquad y(0) = 2,\ y'(0) = 12.$$

(8) (4.5 marks) The current I in an LC series circuit is governed by the initial value problem

$$I''(t) + 4I(t) = g(t), \qquad I(0) = 0,\ I'(0) = 0,$$

where

$$g(t) = \begin{cases} 1 & 0 \le t < 1, \\ -1 & 1 \le t < 2, \\ 0 & 2 \le t. \end{cases}$$

a) Determine the current as a function of time t.

b) Is the current a continuous function of time? Check $I(t)$ for continuity at the points $t = 1$ and $t = 2$ where the nonhomogeneous term is discontinuous.

(9) (3.5 marks) Use the Laplace transform to solve the initial value problem

$$y'' + 2y' = \delta(t-1), \qquad y(0) = 0,\ y'(0) = 1$$