DIFFERENTIAL EQUATIONS, H23, FINAL EXAMINATION

(1) (3 marks) Find the charge $q(t)$ on the capacitor in an LRC series circuit if $L = 0.25H$, $R = 10\Omega$, $C = 0.001F$ and the initial conditions are such that $q(0) = 3C$ and there is no current initially. Express the solution as a single trigonometric function with a phase shift and sketch the graph. Is the circuit underdamped, critically damped or overdamped?

(2) (4 marks) A spring is stretched $6cm$ by a force of $3N$. A mass of $2kg$ is hung from the spring and is also attached to a viscous damper.
i) What should be the value of the damping coefficient γ so that the system is critically damped?
ii) Assuming the system is critically damped determine the motion of the system with initial conditions $u(0) = 0$ and $u'(0) = 2cm/s$. Draw a graph of the solution.
iii) For a spring-mass system what happens to the ratio of the quasi frequency to the natural frequency if the damping coefficient γ changes in such a way that the system is underdamped but approaches critical damping?

(3) (4 marks) The motion of a mechanical system driven by a periodic external force can be modelled as a initial value problem

$$u'' + \omega^2 u = F_0 \sin \gamma t, \quad u(0) = 0, \ u'(0) = 0,$$

where F_0, ω and γ are constants and $\gamma \neq \omega$. Solve the initial value problem. The result should be a function $u(t)$ with the constants F_0, ω, γ as parameters in the function.

Evaluate the limit

$$\lim_{\gamma \to \omega} u(t)$$

with the use of L'Hospitals Rule. In this limit what happens to the amplitude of the oscillations for large t?

(4) (4 marks) Find the fundamental matrix $\Phi(t)$ satisfying $\Phi(0) = \mathbf{I}$ for the following system

$$\begin{aligned} y_1' &= -3y_1 + y_2 \\ y_2' &= 2y_1 - 4y_2 \end{aligned}$$

Use this fundamental matrix to solve this system of DE's with initial conditions $y_1(0) = 1, y_2(0) = -1$.

(5) (4 marks) Determine the general solution of the nonhomogeneous system of DE's

$$\begin{aligned} y_1' &= -3y_1 + y_2 + 3t \\ y_2' &= 2y_1 - 4y_2 + e^{-t} \end{aligned}$$

Notice that the coefficient system is the same as in Problem 2, so you can reuse the pertinent quantities computed there.

(6) (4 marks) Solve the initial value problem

$$\begin{array}{rcl} y_1' & = & 3y_1 - 18y_2 \\ y_2' & = & 2y_1 - 9y_2 \end{array}, \qquad y_1(0) = -1,\ y_2(0) = -1.$$

(7) (4 marks) Consider the system of DE's depending on a parameter α

$$\begin{aligned} y_1' &= 5y_1 + 3y_2 \\ y_2' &= \alpha y_1 + 5y_2 \end{aligned}$$

a) Determine the critical values of α where the qualitative nature of the phase portrait for the system changes.
b) The critical values split the real axis into intervals. Draw a phase portait for the system for α in each of these intervals.

(8) (3 marks) Use Laplace transform to solve the IVP

$$y' + y = f(t),\ y(0) = 5,$$

where

$$f(t) = \begin{cases} 0, & 0 \leq t \leq \pi \\ 3\cos t, & t \geq \pi \end{cases}$$

(9) (3 marks) Use Laplace transform to solve the IVP

$$y' + 3y = 13\sin 2t, \qquad y(0) = 6$$

(10) (4 marks) Use Laplace transform to solve the IVP

$$y'' + y = 2\delta(t - 2\pi), \qquad y(0) = 1,\ y'(0) = 0$$

Next, sketch the graph at the solution. Are the solution, its first and second derivatives continuous on $[0, \infty)$? List the functions which are discontinuous and the points of discontinuity.