DISCRETE MATHEMATICS, A24, TEST 2

Name: ________________________ Student number____________________

(1) (2 marks) On an input of size n a certain computer algorithm executes two times as many operations as on input of size $n-1$ plus additional 2^n operations. When the algorithm is run on inputs of size $n = 1$, it executes 8 operations. Set up a recurrence relation for the number of operations on input of size n.

Using iteration derive a formula for the number of operations the algorithm runs when it is executed on input of size n. Prove your formula by induction.

(2) (2 marks) Find a closed-form solution for the Lucas recurrence relation

$$L(1) = 1, \quad L(2) = 3$$
$$L(n) = L(n-1) + L(n-2), \quad n \geq 3$$

(3) (2 marks) Prove or disprove that for any sets A, B, C and D

$$(A \backslash B) \times (C \backslash D) = (A \times C) \backslash (B \times D).$$

(4) (2 marks) Use Boolean algebra identities to prove the following formula for any sets A and B:

$$(A \cap B) \backslash (B \cap C) = (A \cap B) \backslash C$$

Cite the relevant identity at each step.

(5) (2 marks) Prove or disprove that for any two sets A and B

$$\mathcal{P}(A \backslash B) = \mathcal{P}(A) \backslash \mathcal{P}(B)$$

(6) (2 marks) These days we have AIs (neural networks) which describe AIs (including in some cases themsleves). Formulate a paradox in the style or Cantor and Russell in this context. Explain precisely how your setup generates a paradox.

(7) (2 marks) Prove that if A, B and C are any countable infinite sets, then $A \times B \times C$ is countable.

(8) (2 marks) Using a Hilbert Hotel type of argument prove that the union of three countable sets is countable.

(9) (2.5 marks) Consider a poker hands of 5 cards selected from a standard deck of 52 cards.
  i) How many poker hands have two Kings?
  ii) How many poker hands have only Diamonds?
  iii) How many poker hands have only red cards or only face cards?

(10) (2.5 marks) Given a set of 40 distinct integers, show that there must be two whose sum or difference is divisible by 75.

(11) (2.5 marks) How many integers between 1 and 500 (inclusive) are cube free. An integer , n is called cube free if it does not have a divisor of the form k^3 where $k > 1$ is an integer.

(12) (2.5 marks) Using Elisabeth's public key $n = 187$, $s = 13$ her partner in crime has send a message containing the passcode for the bitcoin account where the millions in ill-gained profits are held, $E = 26$. Working for a top-secret government agency you must figure out how the first component of Elisabeth's public key factorizes into two prime numbers. Determine the factorization and then decode the message E to recover the bitcoin passcode.