

Linear Algebra — Class Exercise 10

Inverses at Work and the Inversion Algorithm

1. One Inverse, Several Systems. Let

$$A = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}, \quad \mathbf{b}_1 = \begin{bmatrix} 11 \\ 8 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{b}_3 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

- (a) Find A^{-1} and verify it.
- (b) Solve the three systems $A\mathbf{x} = \mathbf{b}_1$, $A\mathbf{x} = \mathbf{b}_2$ and $A\mathbf{x} = \mathbf{b}_3$ with a *single* matrix product, by placing $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$ side by side as the columns of one matrix.
- (c) Check each of the three solutions in the original system.
- (d) Without solving anything, explain why $A\mathbf{x} = \mathbf{b}$ has exactly one solution for *every* $\mathbf{b} \in \mathbb{R}^2$.

Solution.

- (a) $\det(A) = 3(3) - 4(2) = 9 - 8 = 1 \neq 0$, so A is invertible. Swap the diagonal entries, negate the off-diagonal entries, divide by 1:

$$A^{-1} = \begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix}.$$

$$\text{Check: } AA^{-1} = \begin{bmatrix} 3(3) + 4(-2) & 3(-4) + 4(3) \\ 2(3) + 3(-2) & 2(-4) + 3(3) \end{bmatrix} = \begin{bmatrix} 9 - 8 & -12 + 12 \\ 6 - 6 & -8 + 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

- (b) Since A is invertible, each system has the unique solution $\mathbf{x}_j = A^{-1}\mathbf{b}_j$. Doing all three at once:

$$A^{-1} \begin{bmatrix} 11 & 1 & 2 \\ 8 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 11 & 1 & 2 \\ 8 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 33 - 32 & 3 - 4 & 6 - 4 \\ -22 + 24 & -2 + 3 & -4 + 3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & -1 \end{bmatrix}.$$

Column j of the result solves the system with right-hand side $\mathbf{b}_j$:

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$

- (c) $A\mathbf{x}_1 = \begin{bmatrix} 3+8 \\ 2+6 \end{bmatrix} = \begin{bmatrix} 11 \\ 8 \end{bmatrix} = \mathbf{b}_1 \checkmark, \quad A\mathbf{x}_2 = \begin{bmatrix} -3+4 \\ -2+3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \mathbf{b}_2 \checkmark, \quad A\mathbf{x}_3 = \begin{bmatrix} 6-4 \\ 4-3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \mathbf{b}_3 \checkmark.$
- (d) A is invertible. For any $\mathbf{b}$, the vector $\mathbf{x} = A^{-1}\mathbf{b}$ is a solution, because $A(A^{-1}\mathbf{b}) = (AA^{-1})\mathbf{b} = \mathbf{b}$. It is the only one: if $A\mathbf{x} = \mathbf{b}$, left-multiplying by A^{-1} forces $\mathbf{x} = A^{-1}\mathbf{b}$. So there are no free variables and no inconsistent right-hand sides, whatever $\mathbf{b}$ is.

2. Solving for a Matrix Between A and A^{-1} . Let

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}.$$

- (a) Solve the equation $AXA^{-1} = B$ for X in terms of A and B . Say at every step whether you left-multiply or right-multiply.
- (b) Find A^{-1} , and then compute X for the given matrices.

- (c) Check your answer by computing AXA^{-1} .
- (d) A student claims that “ A and A^{-1} cancel, so $X = B$ ”. Show that this is false for the given matrices, and explain where the reasoning goes wrong.

Solution.

- (a) A stands immediately to the left of X , so **left**-multiply both sides by A^{-1} :

$$A^{-1}AXA^{-1} = A^{-1}B \implies XA^{-1} = A^{-1}B.$$

A^{-1} stands immediately to the right of X ; the matrix that removes it is $(A^{-1})^{-1} = A$, so **right**-multiply both sides by A :

$$XA^{-1}A = A^{-1}BA \implies X = A^{-1}BA.$$

- (b) $\det(A) = 4 - 3 = 1$, so $A^{-1} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$. Multiply in two steps, keeping the order A^{-1} , B , A :

$$A^{-1}B = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 2-6 & 0+3 \\ -1+4 & 0-2 \end{bmatrix} = \begin{bmatrix} -4 & 3 \\ 3 & -2 \end{bmatrix},$$

$$X = (A^{-1}B)A = \begin{bmatrix} -4 & 3 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} -8+3 & -12+6 \\ 6-2 & 9-4 \end{bmatrix} = \begin{bmatrix} -5 & -6 \\ 4 & 5 \end{bmatrix}.$$

- (c)

$$AX = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -5 & -6 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} -10+12 & -12+15 \\ -5+8 & -6+10 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix},$$

$$AXA^{-1} = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 4-3 & -6+6 \\ 6-4 & -9+8 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} = B \checkmark.$$

- (d) $X = \begin{bmatrix} -5 & -6 \\ 4 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}$ are different matrices, so $X = B$ is false.

A matrix cancels only with its own inverse when the two stand *next to each other*. In AXA^{-1} , the matrix X stands between A and A^{-1} , and since matrix multiplication is not commutative, X cannot be moved out of the way. The student's step would require $AXA^{-1} = AA^{-1}X$, i.e.

$XA^{-1} = A^{-1}X$, which holds only when X and A commute. Here they do not: $AX = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$ while

$$XA = \begin{bmatrix} -16 & -27 \\ 13 & 22 \end{bmatrix}.$$

3. Inverting a 3×3 Matrix. Let

$$C = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 2 & 1 & -1 \end{bmatrix}.$$

- Find C^{-1} by row-reducing the block matrix $[C \mid I]$. Name every row operation you use, and separate the forward phase (zeros below the pivots) from the backward phase (zeros above the pivots).
- At which point of the reduction can you be sure that C is invertible? Why?
- Verify your answer by computing CC^{-1} .

Solution.

- Forward phase.** The first column already has a 1 on top and a 0 in row 2; only row 3 needs clearing. Every operation is applied to all six columns.

$$\begin{aligned} & \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 2 & 1 & -1 & 0 & 0 & 1 \end{array} \right] \xrightarrow{R_3 \rightarrow R_3 - 2R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & 1 & 3 & -2 & 0 & 1 \end{array} \right] \\ & \xrightarrow{R_3 \rightarrow R_3 - R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & 0 & -1 & -2 & -1 & 1 \end{array} \right] \xrightarrow{R_3 \rightarrow -R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & 1 & -1 \end{array} \right]. \end{aligned}$$

Backward phase. Clear above the third pivot. Column 2 has nothing above its pivot, so this finishes the reduction.

$$\begin{aligned} & \xrightarrow[R_1 \rightarrow R_1 + 2R_3]{R_2 \rightarrow R_2 - 4R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 5 & 2 & -2 \\ 0 & 1 & 0 & -8 & -3 & 4 \\ 0 & 0 & 1 & 2 & 1 & -1 \end{array} \right]. \end{aligned}$$

(Row 2: $(0, 1, 4 \mid 0, 1, 0) - 4(0, 0, 1 \mid 2, 1, -1) = (0, 1, 0 \mid -8, -3, 4)$. Row 1: $(1, 0, -2 \mid 1, 0, 0) + 2(0, 0, 1 \mid 2, 1, -1) = (1, 0, 0 \mid 5, 2, -2)$.) Therefore

$$C^{-1} = \begin{bmatrix} 5 & 2 & -2 \\ -8 & -3 & 4 \\ 2 & 1 & -1 \end{bmatrix}.$$

- At the end of the forward phase. The left half then has three pivots in a 3×3 matrix, so $\text{rank}(C) = 3$ and the reduced row echelon form of C must be I_3 . The backward phase cannot fail from there; it only clears entries above the pivots. Had a row of zeros appeared in the left half instead, C would have been singular.
- Row i of C against column j of C^{-1} :

$$CC^{-1} = \begin{bmatrix} 5+0-4 & 2+0-2 & -2+0+2 \\ 0-8+8 & 0-3+4 & 0+4-4 \\ 10-8-2 & 4-3-1 & -4+4+1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \checkmark.$$

For a square matrix, $CC^{-1} = I$ in one order is enough: it forces $C^{-1}C = I$ as well.