

Linear Algebra — Class Exercise 11

The Inversion Algorithm: Singular Matrices and Equivalent Statements

1. Show that $C = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 1 & 4 \\ 5 & -1 & 8 \end{bmatrix}$ is singular, and read off from the right half a combination of its rows that gives zero.

Solution. Row-reduce $[C \mid I]$:

$$\begin{aligned} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 3 & 1 & 4 & 0 & 1 & 0 \\ 5 & -1 & 8 & 0 & 0 & 1 \end{array} \right] &\xrightarrow[R_3 \rightarrow R_3 - 5R_1]{R_2 \rightarrow R_2 - 3R_1} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 4 & -2 & -3 & 1 & 0 \\ 0 & 4 & -2 & -5 & 0 & 1 \end{array} \right] \\ &\xrightarrow{R_3 \rightarrow R_3 - R_2} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 4 & -2 & -3 & 1 & 0 \\ 0 & 0 & 0 & -2 & -1 & 1 \end{array} \right]. \end{aligned}$$

The left half has a row of zeros, so it cannot become I_3 : C is singular, with $\text{rank}(C) = 2$.

The right half of the zero row records what was done to the rows of C : $-2(\text{row } 1) - (\text{row } 2) + (\text{row } 3) = \mathbf{0}$, i.e. $\text{row } 3 = 2(\text{row } 1) + \text{row } 2$. Check: $2(1, -1, 2) + (3, 1, 4) = (5, -1, 8) \checkmark$.

2. Show that $M = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & c \\ 0 & c & 1 \end{bmatrix}$ is invertible for *every* real number c .

Solution. Reduce the left half (the right half is not needed to decide invertibility). $R_2 \rightarrow R_2 - R_1$ gives the row $(0, -1, c)$; then $R_3 \rightarrow R_3 + cR_2$ gives

$$(0, c, 1) + c(0, -1, c) = (0, 0, 1 + c^2).$$

So the forward phase ends with $\begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & c \\ 0 & 0 & 1 + c^2 \end{bmatrix}$. Since $1 + c^2 \geq 1 > 0$ for every real c , the third pivot is never zero. There are always three pivots, so $\text{rank}(M) = 3$ and M is invertible, whatever c is.

3. Solve $AX = B$ by reducing $[A \mid B]$, where $A = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 3 \end{bmatrix}$.

Solution. Reduce $[A \mid B]$, choosing operations that avoid fractions:

$$\begin{aligned} \left[\begin{array}{cc|ccc} 3 & 1 & 1 & 0 & 2 \\ 5 & 2 & 2 & 1 & 3 \end{array} \right] &\xrightarrow{R_2 \rightarrow R_2 - R_1} \left[\begin{array}{cc|ccc} 3 & 1 & 1 & 0 & 2 \\ 2 & 1 & 1 & 1 & 1 \end{array} \right] \\ &\xrightarrow{R_1 \rightarrow R_1 - R_2} \left[\begin{array}{cc|ccc} 1 & 0 & 0 & -1 & 1 \\ 2 & 1 & 1 & 1 & 1 \end{array} \right] \xrightarrow{R_2 \rightarrow R_2 - 2R_1} \left[\begin{array}{cc|ccc} 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & 1 & 3 & -1 \end{array} \right]. \end{aligned}$$

The left half is I , so the right half is $A^{-1}B$:

$$X = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 3 & -1 \end{bmatrix}.$$

Check: $AX = \begin{bmatrix} 0+1 & -3+3 & 3-1 \\ 0+2 & -5+6 & 5-2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 3 \end{bmatrix} = B \checkmark$.

4. True or false? Justify, or give a counterexample.

(a) If A is 3×3 and some system $A\mathbf{x} = \mathbf{b}$ has two different solutions, then A is singular.

(b) If A is square and $A\mathbf{x} = \mathbf{b}$ is consistent for one particular $\mathbf{b}$, then A is invertible.

Solution.

(a) **True.** If A were invertible, then $A\mathbf{x} = \mathbf{b}$ would have exactly one solution for every $\mathbf{b}$ (namely $\mathbf{x} = A^{-1}\mathbf{b}$). Two different solutions are therefore impossible, so A is singular.

(b) **False.** Take $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. The system is consistent ($\mathbf{x} = (1, 0)$ is a solution), but A has a zero row, so it is singular. Invertibility needs the system to be consistent for *every* $\mathbf{b}$, not just one.