Linear Algebra — Class Exercise 12

Cofactor Expansion and Row Reduction of Determinants

1. Let $P = \begin{bmatrix} 2 & -1 & 4 \\ 3 & 5 & 0 \\ -2 & 1 & 6 \end{bmatrix}$. Compute $\det(P)$ along the cheapest row and also along the cheapest column. Which lines did you choose, and why? Do the two answers agree?

   Solution. Every line of P with a zero saves one minor. Row 2 is $(3, 5, 0)$ and column 3 is $(4, 0, 6)$; both contain a single zero, so each costs two minors.

   Row 2, signs $-, +, -$:

   $$\det(P) = -3M_{21} + 5M_{22} - 0 \cdot M_{23},$$

   with $M_{21} = (-1)(6) - (4)(1) = -10$ and $M_{22} = (2)(6) - (4)(-2) = 20$. So

   $$\det(P) = -3(-10) + 5(20) = 30 + 100 = 130.$$

   Column 3, signs $+, -, +$:

   $$\det(P) = 4M_{13} - 0 \cdot M_{23} + 6M_{33},$$

   with $M_{13} = (3)(1) - (5)(-2) = 13$ and $M_{33} = (2)(5) - (-1)(3) = 13$. So

   $$\det(P) = 4(13) + 6(13) = 52 + 78 = 130.$$

   The two answers agree: $\det(P) = 130$.

2. Compute $\det \begin{bmatrix} 2 & 1 & 0 & 3 \\ 4 & -1 & 5 & 1 \\ 0 & 3 & 0 & 0 \\ 1 & 2 & 6 & -2 \end{bmatrix}$, saying which line you chose and why.

   Solution. Row 3 is $(0, 3, 0, 0)$, so only one term of the expansion survives. The entry $a_{32} = 3$ sits on a $-$ square of the checkerboard, so $\det = -3M_{32}$. The minor M_{32} (delete row 3 and column 2) has rows $(2, 0, 3)$, $(4, 5, 1)$, $(1, 6, -2)$. Expand it along its row 1, which has a zero:

   $$M_{32} = 2\begin{vmatrix} 5 & 1 \\ 6 & -2 \end{vmatrix} - 0 + 3\begin{vmatrix} 4 & 5 \\ 1 & 6 \end{vmatrix} = 2(-10 - 6) + 3(24 - 5) = -32 + 57 = 25.$$

   Therefore $\det = -3(25) = -75$.

3. Find the false step and give the correct value. A student computes $\begin{vmatrix} 2 & 0 & 1 \\ 1 & 3 & -1 \\ 4 & 1 & 2 \end{vmatrix}$ along column 2:

   Step 1. Column 2 is $(0, 3, 1)$.

   Step 2. $\det = 0\begin{vmatrix} 1 & -1 \\ 4 & 2 \end{vmatrix} - 3\begin{vmatrix} 2 & 1 \\ 4 & 2 \end{vmatrix} + 1\begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix}$.

   Step 3. $= 0 - 3(0) + 1(-3) = -3$.

   Solution. Step 2 is the false step: it uses the signs of row 1 $(+, -, +)$ on column 2. Down column 2 the checkerboard reads $-, +, -$, so

$$\det = -0 \cdot M_{12} + 3M_{22} - 1 \cdot M_{32} = 3\begin{vmatrix} 2 & 1 \\ 4 & 2 \end{vmatrix} - \begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} = 3(4-4) - (-2-1) = 3.$$

   The arithmetic in Step 3 was right; because the signs were wrong, the answer came out exactly negated. The correct value is 3.

4. Evaluate by reduction: $\begin{vmatrix} 1 & -2 & 3 \\ 2 & -3 & 8 \\ -3 & 5 & -7 \end{vmatrix}$.

   Solution. Use only row replacements, which do not change the determinant. $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 + 3R_1$ give rows $(0, 1, 2)$ and $(0, -1, 2)$; then $R_3 \to R_3 + R_2$ gives $(0, 0, 4)$:

$$\begin{vmatrix} 1 & -2 & 3 \\ 2 & -3 & 8 \\ -3 & 5 & -7 \end{vmatrix} = \begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 2 \\ 0 & -1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{vmatrix} = 1 \cdot 1 \cdot 4 = 4.$$