Linear Algebra — Class Exercise 13

Properties of Determinants

1. **Determinants Without Matrices.** A and B are 4×4 matrices with $\det(A) = 2$ and $\det(B) = -3$. Find

   (a) $\det\big(A^{-1}(3B)^T\big)$,

   (b) $\det(A^3B^2)$,

   (c) $\det(-B)$.

   Name the rule you use at each step. Do you need to know any entry of A or B?

   Solution. No entry of either matrix is needed; only $n = 4$, $\det(A)$ and $\det(B)$.

   (a) Work from the outside in:

   $$\det\big(A^{-1}(3B)^T\big) = \det(A^{-1})\det\big((3B)^T\big) = \frac{1}{\det A} \cdot \det(3B) = \frac{1}{2} \cdot 3^4 \det(B) = \frac{1}{2} \cdot 81 \cdot (-3) = -\frac{243}{2}.$$

   (product rule; inverse rule and $\det(M^T) = \det(M)$; scalar rule with $n = 4$.)

   (b) $\det(A^3B^2) = \det(A)^3 \det(B)^2 = 2^3 \cdot (-3)^2 = 8 \cdot 9 = 72$.

   (c) $\det(-B) = (-1)^4 \det(B) = -3$. Since $n = 4$ is even, the sign does not change.

2. **The Scalar Rule.** Three students make the following claims. In each case find the mistake and give the correct value.

   (a) A is 4×4 with $\det(A) = 3$. "Then $\det(-A) = -3$."

   (b) A is 3×3 with $\det(A) = 5$. "Then $\det\big((2A)^{-1}\big) = 2^3/5 = 8/5$."

   (c) A is 4×4 with $\det(A) = -2$. "Then $\det\big((-2A)^{-1}\big) = 1/(-2 \cdot (-2)) = 1/4$."

   Solution.

   (a) The sign of -1 is raised to the power n, not to the power 1: $\det(-A) = (-1)^4 \det(A) = 3$. For even n the determinant does not change.

   (b) The 2 is *inside* the inverse. Work from the outside in: $\det\big((2A)^{-1}\big) = \dfrac{1}{\det(2A)} = \dfrac{1}{2^3 \cdot 5} = \dfrac{1}{40}$. The factor 2^n belongs in the denominator, not the numerator.

   (c) The scalar -2 multiplies all four rows, so $\det(-2A) = (-2)^4 \det(A) = 16 \cdot (-2) = -32$, not $(-2) \cdot (-2)$. Hence $\det\big((-2A)^{-1}\big) = \dfrac{1}{-32} = -\dfrac{1}{32}$.

3. **What $\det = 0$ Does Not Tell You.** Let $N = \begin{bmatrix} 1 & 2 & 1 \\ 2 & k & 2 \\ 1 & 2 & k \end{bmatrix}$.

   (a) Find the values of k for which N is singular.

(b) A student says: "At $k = 1$, $\det(N) = 0$, so $N\mathbf{x} = \mathbf{b}$ has no solution." Show that this is false by solving $N\mathbf{x} = \mathbf{b}$ for $\mathbf{b} = (1, 2, 1)^T$, and show that for $\mathbf{b} = (1, 2, 2)^T$ there really is no solution.

(c) Repeat at $k = 4$ with $\mathbf{b} = (1, 2, 3)^T$ and $\mathbf{b} = (1, 0, 3)^T$.

(d) What does $\det(N) = 0$ tell you about $N\mathbf{x} = \mathbf{b}$, and what must you do to find out more?

Solution.

(a) $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - R_1$ give rows $(0, k-4, 0)$ and $(0, 0, k-1)$, so $\det(N) = (k-4)(k-1)$. N is singular for $k = 1$ and $k = 4$.

(b) At $k = 1$ reduce $[\, N \mid \mathbf{b} \,]$. Rows R_1 and R_3 of N are equal, and $R_2 \to R_2 - 2R_1$ gives $(0, -3, 0)$.

For $\mathbf{b} = (1, 2, 1)^T$: the rows become $(1, 2, 1 \mid 1)$, $(0, -3, 0 \mid 0)$, $(0, 0, 0 \mid 0)$. Consistent: $x_2 = 0$, $x_1 + x_3 = 1$ with x_3 free, so there are *infinitely many* solutions, e.g. $(1, 0, 0)^T$ and $(0, 0, 1)^T$.

For $\mathbf{b} = (1, 2, 2)^T$: after $R_3 \to R_3 - R_1$ the last row is $(0, 0, 0 \mid 1)$, i.e. $0 = 1$. *No solution.*

(c) At $k = 4$: $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - R_1$ turn N into rows $(1, 2, 1)$, $(0, 0, 0)$, $(0, 0, 3)$; the zero row needs $b_2 - 2b_1 = 0$.

For $\mathbf{b} = (1, 2, 3)^T$: $b_2 - 2b_1 = 0$, and the rows read $(1, 2, 1 \mid 1)$, $(0, 0, 0 \mid 0)$, $(0, 0, 3 \mid 2)$. Consistent: $x_3 = \frac{2}{3}$, $x_1 + 2x_2 = \frac{1}{3}$, with x_2 free: *infinitely many* solutions.

For $\mathbf{b} = (1, 0, 3)^T$: $b_2 - 2b_1 = -2 \neq 0$, the zero row becomes $(0, 0, 0 \mid -2)$. *No solution.*

(d) $\det(N) \neq 0$ means exactly one solution for every $\mathbf{b}$. But $\det(N) = 0$ only says "not exactly one": depending on $\mathbf{b}$ there may be no solution or infinitely many, as parts (b) and (c) show. The determinant finds the critical values of k; to classify each one, go back to elimination on $[\, N \mid \mathbf{b} \,]$.