# Linear Algebra — Class Exercise 14

The Adjugate and Cramer's Rule

1. **Inverse by the Adjugate.** Let $A = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 3 & -1 \\ 0 & 2 & 1 \end{bmatrix}$.

   (a) Compute $\det(A)$ from the cofactors of row 1.

   (b) Compute the other six cofactors, and write down $\operatorname{cof}(A)$ and $\operatorname{adj}(A)$.

   (c) Write $A^{-1}$, and check one row of $A \operatorname{adj}(A)$.

   **Solution.**

   (a) Signs $+, -, +$ along row 1:

   $$C_{11} = +\begin{vmatrix} 3 & -1 \\ 2 & 1 \end{vmatrix} = 5, \quad C_{12} = -\begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = -1, \quad C_{13} = +\begin{vmatrix} 1 & 3 \\ 0 & 2 \end{vmatrix} = 2,$$

   so $\det(A) = 2(5) + 0(-1) + 1(2) = 12$. It is non-zero, so $A$ is invertible.

   (b) Row 2 has signs $-, +, -$ and row 3 has signs $+, -, +$:

   $$C_{21} = -\begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix} = 2, \quad C_{22} = +\begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} = 2, \quad C_{23} = -\begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = -4,$$

   $$C_{31} = +\begin{vmatrix} 0 & 1 \\ 3 & -1 \end{vmatrix} = -3, \quad C_{32} = -\begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} = 3, \quad C_{33} = +\begin{vmatrix} 2 & 0 \\ 1 & 3 \end{vmatrix} = 6.$$

   Each cofactor goes in its own position, and then the matrix is transposed:

   $$\operatorname{cof}(A) = \begin{bmatrix} 5 & -1 & 2 \\ 2 & 2 & -4 \\ -3 & 3 & 6 \end{bmatrix}, \qquad \operatorname{adj}(A) = \operatorname{cof}(A)^T = \begin{bmatrix} 5 & 2 & -3 \\ -1 & 2 & 3 \\ 2 & -4 & 6 \end{bmatrix}.$$

   (c)

   $$A^{-1} = \frac{1}{12}\begin{bmatrix} 5 & 2 & -3 \\ -1 & 2 & 3 \\ 2 & -4 & 6 \end{bmatrix}.$$

   Check: row 1 of $A$, $(2, 0, 1)$, against the three columns of $\operatorname{adj}(A)$ gives $10+0+2 = 12$, $4+0-4 = 0$, $-6+0+6 = 0$. That is row 1 of $12\,I$, as the adjugate identity predicts.

2. **A $3 \times 3$ System by Cramer's Rule.** Solve

   $$\begin{aligned} x + y + 2z &= 9 \\ 2x - y + z &= 3 \\ x + 3y - z &= 4 \end{aligned}$$

   by Cramer's rule: compute $\det(A)$, $\det(A_1)$, $\det(A_2)$ and $\det(A_3)$, and then check your solution in all three equations.

**Solution.** The coefficient matrix is $A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & -1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$ and $\mathbf{b} = (9, 3, 4)^T$. Expanding along row 1:

$$\det(A) = 1(1-3) - 1(-2-1) + 2(6+1) = -2 + 3 + 14 = 15 \neq 0,$$

so there is exactly one solution. Replace column $i$ by $\mathbf{b}$:

$$\det(A_1) = \begin{vmatrix} 9 & 1 & 2 \\ 3 & -1 & 1 \\ 4 & 3 & -1 \end{vmatrix} = 9(1-3) - 1(-3-4) + 2(9+4) = -18 + 7 + 26 = 15,$$

$$\det(A_2) = \begin{vmatrix} 1 & 9 & 2 \\ 2 & 3 & 1 \\ 1 & 4 & -1 \end{vmatrix} = 1(-3-4) - 9(-2-1) + 2(8-3) = -7 + 27 + 10 = 30,$$

$$\det(A_3) = \begin{vmatrix} 1 & 1 & 9 \\ 2 & -1 & 3 \\ 1 & 3 & 4 \end{vmatrix} = 1(-4-9) - 1(8-3) + 9(6+1) = -13 - 5 + 63 = 45.$$

Therefore

$$x = \frac{15}{15} = 1, \qquad y = \frac{30}{15} = 2, \qquad z = \frac{45}{15} = 3.$$

Check: $1 + 2 + 6 = 9$, $2 - 2 + 3 = 3$, $1 + 6 - 3 = 4$ ✓.

3. **A Formula, Not a Number.** Consider the system

$$\begin{aligned} x + ky &= 2 \\ kx + 4y &= 1 \end{aligned}$$

(a) For which values of $k$ does the system have a unique solution?

(b) For those $k$, use Cramer's rule to find $x$ and $y$ as expressions in $k$.

(c) For each value of $k$ found in (a) where the solution is not unique, use elimination to decide whether the system has no solution or infinitely many.

**Solution.**

(a) $\det(A) = \begin{vmatrix} 1 & k \\ k & 4 \end{vmatrix} = 4 - k^2 = (2-k)(2+k)$. The solution is unique exactly when $\det(A) \neq 0$, i.e. for $k \neq 2$ and $k \neq -2$.

(b) Put $\mathbf{b} = (2, 1)^T$ in column 1 for $x$, and in column 2 for $y$:

$$x = \frac{\begin{vmatrix} 2 & k \\ 1 & 4 \end{vmatrix}}{4 - k^2} = \frac{8 - k}{4 - k^2}, \qquad y = \frac{\begin{vmatrix} 1 & 2 \\ k & 1 \end{vmatrix}}{4 - k^2} = \frac{1 - 2k}{4 - k^2}.$$

(c) The values found in (a) are $k = 2$ and $k = -2$. At $k = 2$ the system is $x + 2y = 2$, $2x + 4y = 1$. Then $R_2 \to R_2 - 2R_1$ gives $0 = -3$: no solution.

At $k = -2$ the system is $x - 2y = 2$, $-2x + 4y = 1$. Then $R_2 \to R_2 + 2R_1$ gives $0 = 5$: no solution.

So the system is inconsistent at both critical values. In the formulas this shows up as a denominator $4 - k^2$ tending to 0 while the numerators $8 - k$ and $1 - 2k$ do not.