Linear Algebra — Class Exercise 15

Vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$

1. For $A(-1,2,5)$ and $B(4,-3,1)$, find $\overrightarrow{AB}$, $\overrightarrow{BA}$ and the distance from A to B.

   Solution. $\overrightarrow{AB} = B - A = (4-(-1),\ -3-2,\ 1-5) = (5,-5,-4)$, and $\overrightarrow{BA} = -\overrightarrow{AB} = (-5,5,4)$. The distance from A to B is the length of $\overrightarrow{AB}$:

   $$\|\overrightarrow{AB}\| = \sqrt{25+25+16} = \sqrt{66}.$$

2. For $\mathbf{u} = (2,-1,3)$ and $\mathbf{v} = (4,1,-5)$, find $3\mathbf{u} - 2\mathbf{v}$, $\|\mathbf{u}+\mathbf{v}\|$, and the vector $\mathbf{w}$ with $\mathbf{u} - 2\mathbf{w} = \mathbf{v}$.

   Solution. $3\mathbf{u} - 2\mathbf{v} = (6,-3,9) - (8,2,-10) = (-2,-5,19)$.

   $\mathbf{u}+\mathbf{v} = (6,0,-2)$, so $\|\mathbf{u}+\mathbf{v}\| = \sqrt{36+0+4} = \sqrt{40} = 2\sqrt{10}$.

   From $\mathbf{u} - 2\mathbf{w} = \mathbf{v}$ we get $2\mathbf{w} = \mathbf{u} - \mathbf{v} = (-2,-2,8)$, so $\mathbf{w} = (-1,-1,4)$. Check: $\mathbf{u} - 2\mathbf{w} = (2+2,\ -1+2,\ 3-8) = (4,1,-5) = \mathbf{v}$ ✓.

3. **(a)** Find k so that $(k,10,-15)$ is parallel to $(4,-4,6)$.
   (b) Find all k with $\|(k,2,k)\| = 6$.

   Solution. **(a)** Compare the other two components: $\dfrac{10}{-4} = -\dfrac{5}{2}$ and $\dfrac{-15}{6} = -\dfrac{5}{2}$. They agree, so the vectors are parallel when the first components have the same ratio: $k = -\dfrac{5}{2}\cdot 4 = -10$. Check: $(-10,10,-15) = -\dfrac{5}{2}(4,-4,6)$ ✓.

   (b) $k^2 + 4 + k^2 = 36$, so $2k^2 = 32$ and $k^2 = 16$. Hence $k = \pm 4$. Both values are answers.

4. $A(-2,4,1)$ and $B(7,-2,10)$. Find the midpoint of AB, and the point P two thirds of the way from A to B.

   Solution. $\overrightarrow{AB} = (9,-6,9)$. The midpoint is $A + \frac{1}{2}\overrightarrow{AB} = (-2+\frac{9}{2},\ 4-3,\ 1+\frac{9}{2}) = \left(\frac{5}{2}, 1, \frac{11}{2}\right)$, the average of the coordinates of A and B.

   $P = A + \frac{2}{3}\overrightarrow{AB} = (-2+6,\ 4-4,\ 1+6) = (4,0,7)$.

5. Three corners of the parallelogram $ABCD$ are $A(1,-2,3)$, $B(4,0,5)$, $C(6,3,4)$. Find D and verify it.

   Solution. In a parallelogram $\overrightarrow{AD} = \overrightarrow{BC}$, so $D = A + \overrightarrow{BC} = A + C - B = (1+6-4,\ -2+3-0,\ 3+4-5) = (3,1,2)$.

   Verify: $\overrightarrow{AD} = (2,3,-1) = \overrightarrow{BC}$, and $\overrightarrow{DC} = (3,2,2) = \overrightarrow{AB}$ ✓.