# Linear Algebra — Class Exercise 1

Systems of Linear Equations

1. Write the augmented matrix of

$$\begin{aligned} 3z - 2y &= 5 + x \\ 4x &= 1 - z \\ y + 2x &= 0 \end{aligned}$$

using the variable order $x$, $y$, $z$.

**Solution.** In standard form the system is

$$\begin{aligned} -x - 2y + 3z &= 5 \\ 4x + 0y + z &= 1 \\ 2x + y + 0z &= 0 \end{aligned}$$

so the augmented matrix is

$$\left[\begin{array}{ccc|c} -1 & -2 & 3 & 5 \\ 4 & 0 & 1 & 1 \\ 2 & 1 & 0 & 0 \end{array}\right].$$

Three things had to happen before any number could be copied: $x$ moved to the left and became $-x$; $z$ moved to the left and became $+z$; and the variables missing from rows 2 and 3 were written in with coefficient 0. Skipping a column instead of writing the zero is the most common way to lose this question.

2. Starting from

$$\left[\begin{array}{ccc|c} 1 & 3 & -2 & 4 \\ 3 & 8 & 1 & 14 \\ 0 & 1 & 5 & -2 \end{array}\right]$$

apply, in this order: $R_2 \to R_2 - 3R_1$, then $R_1 \leftrightarrow R_3$, then $R_2 \to R_2 + R_1$. Write the matrix after each step.

**Solution.** After $R_2 \to R_2 - 3R_1$:

$$\left[\begin{array}{ccc|c} 1 & 3 & -2 & 4 \\ 0 & -1 & 7 & 2 \\ 0 & 1 & 5 & -2 \end{array}\right]$$

After $R_1 \leftrightarrow R_3$:

$$\left[\begin{array}{ccc|c} 0 & 1 & 5 & -2 \\ 0 & -1 & 7 & 2 \\ 1 & 3 & -2 & 4 \end{array}\right]$$

After $R_2 \to R_2 + R_1$:

$$\left[\begin{array}{ccc|c} 0 & 1 & 5 & -2 \\ 0 & 0 & 12 & 0 \\ 1 & 3 & -2 & 4 \end{array}\right]$$

Two things to notice. The third operation adds the *new* $R_1$ — the one the interchange put there — because operations are applied in sequence to the matrix as it stands. And the augmented column takes part in every operation: $2 + (-2) = 0$ in the last step, not 2.

3. Find the false step in this argument, and say why it fails.

   **Claim.** A system of two linear equations in three unknowns always has infinitely many solutions.

   1. Each equation $ax + by + cz = d$ describes a plane in space.
   2. Two distinct planes meet in a line.
   3. A line contains infinitely many points.
   4. So every such system has infinitely many solutions.

   **Solution.** Step 2 is false. Two distinct planes meet in a line only when they are not parallel; distinct parallel planes never meet at all. For example

   $$\begin{aligned} x + y + z &= 1 \\ x + y + z &= 2 \end{aligned}$$

   are two distinct planes with no common point, so that system has *no* solutions rather than infinitely many.

   The step states for all distinct planes something that holds only for non-parallel ones, and steps 3 and 4 then inherit the gap. What decides the matter is not the count of equations but whether they disagree.

4. Give a system of three linear equations in three unknowns which is inconsistent, in which no equation is a multiple of any other, and in which *any two* of the three equations, taken on their own, form a consistent system. Describe its solution set.

   **Solution.** For example

   $$\begin{aligned} x + y + z &= 1 \\ x - y + z &= 3 \\ 2x \qquad + 2z &= 6 \end{aligned}$$

   No one equation is a multiple of another. Any two taken alone have non-parallel normals, so each pair meets in a line and is consistent. But adding the first two gives $2x + 2z = 4$, while the third demands $2x + 2z = 6$. The two cannot both hold, so the solution set is *empty*.

   The condition rules out the easy answer of contradicting one equation directly — something like $x + y + z = 1$ alongside $x + y + z = 2$ — and forces the contradiction to be built out of a *combination* of the others. Geometrically these are three planes with no common point, meeting one another in three parallel lines.