Linear Algebra — Class Exercise 2

Gaussian Elimination

1. Two of these matrices are in row echelon form and two are not. Say which two are not, and for each name the condition it breaks.

$$A = \begin{bmatrix} 1 & -3 & 5 & 0 \\ 0 & 1 & 2 & -4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 4 & -1 \\ 0 & 5 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 1 & 3 & 2 \\ 1 & 0 & -2 & 5 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad D = \begin{bmatrix} 0 & 1 & -6 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Solution. B and C are the two that fail.

B**: condition 2.** The first non-zero entry of row 2 is a 5, not a 1. Scaling by $\frac{1}{5}$ would repair it.

C**: condition 3.** Row 1 leads in column 2, but row 2 leads in column 1 — to the *left* of the row above, so the staircase runs backwards. Interchanging rows 1 and 2 would repair it.

A**: yes.** Leading 1s in columns 1, 2 and 4. Column 3 never receives a pivot, and nothing requires it to: the leading 1s must move strictly right, not by exactly one column each time.

D**: yes.** The first column is entirely zero, which is allowed — column 1 simply is not a pivot column. The leading 1s are in columns 2 and 3, and the zero row is at the bottom.

2. Solve

$$\begin{aligned} x + 2y - z &= 7 \\ 2x + 4y + z &= 8 \\ x + 5y + 2z &= 4 \end{aligned}$$

by reducing the augmented matrix to row echelon form and then back-substituting. Name every row operation.

Solution. The augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 7 \\ 2 & 4 & 1 & 8 \\ 1 & 5 & 2 & 4 \end{array}\right]$$

The first pivot entry is already 1, so clear column 1:

$$\xrightarrow{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 7 \\ 0 & 0 & 3 & -6 \\ 1 & 5 & 2 & 4 \end{array}\right] \xrightarrow{R_3 \to R_3 - R_1} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 7 \\ 0 & 0 & 3 & -6 \\ 0 & 3 & 3 & -3 \end{array}\right]$$

The entry in the second pivot position is now 0, so the algorithm calls for an interchange with a row below whose entry in column 2 is not zero. Row 3 has a 3 there:

$$\xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 7 \\ 0 & 3 & 3 & -3 \\ 0 & 0 & 3 & -6 \end{array}\right] \xrightarrow[R_3 \to \frac{1}{3}R_3]{R_2 \to \frac{1}{3}R_2} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 7 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

Back-substitution, bottom row first:

$$z = -2, \qquad y + (-2) = -1 \;\Rightarrow\; y = 1, \qquad x + 2(1) - (-2) = 7 \;\Rightarrow\; x = 3.$$

So $(x, y, z) = (3, 1, -2)$.

Check in the *original* system: $3 + 2 + 2 = 7$ ✓, $6 + 4 - 2 = 8$ ✓, $3 + 5 - 4 = 4$ ✓.

Note that the interchange was not a matter of taste. Once column 1 was cleared, the $(2, 2)$ entry was zero, and no amount of scaling or adding row 2 to itself could produce a pivot there — the entry had to be fetched from a row below.

3. Reduce

$$\begin{aligned} x + y - 2z &= 1 \\ 2x + 3y - z &= 4 \\ 3x + 4y - 3z &= 8 \end{aligned}$$

to row echelon form and state its solution set.

Solution.

$$\left[\begin{array}{ccc|c} 1 & 1 & -2 & 1 \\ 2 & 3 & -1 & 4 \\ 3 & 4 & -3 & 8 \end{array}\right] \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 1 & -2 & 1 \\ 0 & 1 & 3 & 2 \\ 0 & 1 & 3 & 5 \end{array}\right] \xrightarrow{R_3 \to R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 1 & -2 & 1 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 3 \end{array}\right]$$

The last row reads $0x + 0y + 0z = 3$, that is $0 = 3$, which is false whatever x, y and z are. The system is **inconsistent**: its solution set is *empty*.

Stop here. Scaling that row to $[\,0\;\,0\;\,0 \mid 1\,]$ would make the matrix formally row echelon, but there is nothing left to learn from it — once a row of the form $[\,0\;0\;0 \mid c\,]$ with $c \neq 0$ appears, no further elimination can rescue the system.

Where the contradiction came from is worth a moment. Rows 2 and 3 came out with identical left-hand sides and different right-hand sides: the second and third equations make incompatible demands of the very same combination of the unknowns. Geometrically, three planes with no point common to all of them.