Linear Algebra — Class Exercise 3

Gauss–Jordan and the Reduced Row Echelon Form

1. For each matrix, say whether it is in *reduced row echelon form*, in *row echelon form only*, or in *neither*. When a condition fails, name it and say which entry is at fault.

$$M_1 = \begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 2 \end{bmatrix} \qquad M_2 = \begin{bmatrix} 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad M_3 = \begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

$$M_4 = \begin{bmatrix} 1 & -2 & 4 \\ 0 & 3 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad M_5 = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 5 \end{bmatrix} \qquad M_6 = \begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Solution. Two are reduced, two are echelon only, and two are not echelon at all.

M_1**: reduced.** Leading 1s in columns 1, 2 and 3, each the only non-zero entry in its column. Column 4 carries no pivot, so the -3, the 6 and the 2 are unconstrained — indeed if this were an augmented matrix it would read $x = -3$, $y = 6$, $z = 2$.

M_2**: reduced.** Leading 1s in columns 2 and 4, strictly increasing, each alone in its column. Two things that look wrong are not: column 1 is entirely zero, which is allowed — nothing says the first pivot must be in the first column — and the -4 sits in column 3, which has no pivot.

M_3**: echelon only.** The staircase is correct and every entry below a pivot is zero, so conditions 1–3 hold. But condition 4 fails at the 3 in position $(1,2)$: it lies *above* the leading 1 of row 2. Note that the column-3 pivot is already clean — one bad entry is enough. $R_1 \to R_1 - 3R_2$ repairs it and gives $\left[\begin{smallmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 4 \end{smallmatrix}\right]$.

M_4**: neither.** Condition 2 fails: the first non-zero entry of row 2 is a 3, not a 1. It is not in row echelon form, so asking whether it is *reduced* does not yet arise. $R_2 \to \frac{1}{3}R_2$ would repair it.

M_5**: neither.** Condition 1 fails: the zero row is in the middle, not at the bottom. Watch the repair — $R_2 \leftrightarrow R_3$ gives $\left[\begin{smallmatrix} 1 & 0 & -1 & 2 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{smallmatrix}\right]$, which is echelon but still *not* reduced, because of the -1 above the column-3 pivot. Fixing condition 1 does not fix condition 4.

M_6**: echelon only.** Leading 1s in columns 1, 2 and 4. Condition 4 fails at the 4 in position $(1,4)$, above the leading 1 of row 3. The 2 and the -1 in column 3 are innocent: column 3 has no pivot. This is the easiest offence to miss, because the eye stops checking once it reaches the last column. $R_1 \to R_1 - 4R_3$ repairs it.

2. Solve

$$\begin{aligned} 3x + y + 2z &= 11 \\ 2x + 3y - z &= -2 \\ -2x + y + 4z &= 7 \end{aligned}$$

by Gauss–Jordan, carrying the reduction all the way to reduced row echelon form. Name every row operation, and check your answer in the original system.

Hint. The $(1, 1)$ entry is a 3. Before scaling row 1 by $\frac{1}{3}$ and dragging thirds through the whole problem, look at what row 2 has in that column.

Solution. Scaling R_1 by $\frac{1}{3}$ is legal but expensive, and interchanging does not help — the entries in column 1 are 3, 2 and -2, and none of them is a 1. But $3 - 2 = 1$, so a single replacement manufactures the pivot for nothing:

$$\left[\begin{array}{ccc|c} 3 & 1 & 2 & 11 \\ 2 & 3 & -1 & -2 \\ -2 & 1 & 4 & 7 \end{array}\right] \xrightarrow{R_1 \to R_1 - R_2} \left[\begin{array}{ccc|c} 1 & -2 & 3 & 13 \\ 2 & 3 & -1 & -2 \\ -2 & 1 & 4 & 7 \end{array}\right]$$

Forward phase. Clear column 1:

$$\xrightarrow[R_3 \to R_3 + 2R_1]{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & -2 & 3 & 13 \\ 0 & 7 & -7 & -28 \\ 0 & -3 & 10 & 33 \end{array}\right] \xrightarrow{R_2 \to \frac{1}{7}R_2} \left[\begin{array}{ccc|c} 1 & -2 & 3 & 13 \\ 0 & 1 & -1 & -4 \\ 0 & -3 & 10 & 33 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 + 3R_2} \left[\begin{array}{ccc|c} 1 & -2 & 3 & 13 \\ 0 & 1 & -1 & -4 \\ 0 & 0 & 7 & 21 \end{array}\right] \xrightarrow{R_3 \to \frac{1}{7}R_3} \left[\begin{array}{ccc|c} 1 & -2 & 3 & 13 \\ 0 & 1 & -1 & -4 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

That is row echelon form. **Backward phase**, taking the pivots right to left. The rightmost is in row 3, and both rows above have a non-zero entry in column 3:

$$\xrightarrow[R_2 \to R_2 + R_3]{R_1 \to R_1 - 3R_3} \left[\begin{array}{ccc|c} 1 & -2 & 0 & 4 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right] \xrightarrow{R_1 \to R_1 + 2R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Read the answer off the last column: $(x, y, z) = (2, -1, 3)$. No back-substitution.

Check in the *original* system: $6 - 1 + 6 = 11$ ✓, $4 - 3 - 3 = -2$ ✓, $-4 - 1 + 12 = 7$ ✓.

The whole reduction stayed in integers apart from the two divisions by 7, and both of those came out whole. That is the payoff of the first move: $R_1 \to R_1 - R_2$ is an ordinary replacement operation, and nothing obliges you to reach a leading 1 by scaling when subtracting another row gets you there without fractions.

3. Solve

$$\begin{aligned} x + y + z &= 2 \\ 2x - y + 3z &= -1 \\ x + 2y - z &= 7 \\ 3x \quad\quad + 4z &= 1 \end{aligned}$$

by Gauss–Jordan. Four equations, three unknowns — how many solutions does it have, and what does the row of zeros tell you? Check your answer in the original system.

Solution.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 2 & -1 & 3 & -1 \\ 1 & 2 & -1 & 7 \\ 3 & 0 & 4 & 1 \end{array}\right] \xrightarrow{\substack{R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \\ R_4 \to R_4 - 3R_1}} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -3 & 1 & -5 \\ 0 & 1 & -2 & 5 \\ 0 & -3 & 1 & -5 \end{array}\right]$$

Row 3 already has a 1 in the second pivot position, so interchange rather than scale:

$$\xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & -2 & 5 \\ 0 & -3 & 1 & -5 \\ 0 & -3 & 1 & -5 \end{array}\right] \xrightarrow[R_4 \to R_4 + 3R_2]{R_3 \to R_3 + 3R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & -2 & 5 \\ 0 & 0 & -5 & 10 \\ 0 & 0 & -5 & 10 \end{array}\right]$$

$$\xrightarrow{R_4 \to R_4 - R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & -2 & 5 \\ 0 & 0 & -5 & 10 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_3 \to -\frac{1}{5} R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & -2 & 5 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Backward phase, right to left:

$$\xrightarrow[R_2 \to R_2 + 2R_3]{R_1 \to R_1 - R_3} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1 \to R_1 - R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

So $(x, y, z) = (3, 1, -2)$, and it is the *only* solution.

Check in the original system: $3+1-2 = 2$ ✓, $6-1-6 = -1$ ✓, $3+2+2 = 7$ ✓, $9+0-8 = 1$ ✓.

What the zero row means. It does *not* mean infinitely many solutions. It means one equation was redundant: here the fourth is the sum of the first two, $(1, 1, 1 \mid 2) + (2, -1, 3 \mid -1) = (3, 0, 4 \mid 1)$, so it imposes nothing the others had not already imposed. Four equations, but only three independent ones.

What decides the outcome is the number of pivots, and there are three of them — one in each variable column — so every variable is pinned down. In the language of the next section: $\operatorname{rank}(A) = \operatorname{rank}[\,A \mid \mathbf{b}\,] = 3 = n$, which is exactly the condition for a unique solution. Had the zero row instead come out as $[\,0\ 0\ 0 \mid c\,]$ with $c \neq 0$, the redundant equation would have been *contradicted* rather than merely repeated, and there would have been no solution at all.