# Linear Algebra — Class Exercise 4

Rank, Free Variables and the General Solution

1. Each matrix below is the *reduced row echelon form* of the augmented matrix of a system — no elimination is left to do. For each one state rank$(A)$, state rank$[\,A \mid \mathbf{b}\,]$, say how many solutions the system has, and name the leading and the free variables. Where the solution set is infinite, write the general solution in vector form and say how many parameters the rank criterion predicted.

$$N_1 = \left[\begin{array}{cccc|c} 1 & -2 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right] \qquad N_2 = \left[\begin{array}{ccc|c} 1 & 0 & 5 & 2 \\ 0 & 1 & -3 & 7 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

$$N_3 = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad N_4 = \left[\begin{array}{cccc|c} 0 & 1 & 0 & -3 & 2 \\ 0 & 0 & 1 & 2 & -4 \end{array}\right]$$

*Hint.* Count the leading 1s first, then ask whether any of them stands in the last column. Both ranks come off the same matrix.

2. Let $c$ be a number, and consider

$$\begin{aligned} x_1 + 2x_2 - x_3 + 3x_4 &= 2 \\ 2x_1 + 4x_2 - x_3 + 8x_4 &= 7 \\ 3x_1 + 6x_2 - 4x_3 + 7x_4 &= c \end{aligned}$$

Reduce the augmented matrix by Gauss–Jordan and decide for which $c$ the system is consistent, giving both ranks in each case. For the value of $c$ that works, say which variables lead and which are free and write the general solution in vector form, checking it in the original system at a choice of the parameters that is not all zeros. Is there any $c$ at all for which the system has exactly one solution?

*Hint.* The letter sits in the constant column and the row operations ignore the bar, so carry $c$ along untouched — the coefficients decide the arithmetic and $c$ only comes to matter at the last row.

3. Consider the homogeneous system

$$\begin{aligned} x_1 + 2x_2 - x_3 + x_4 &= 0 \\ 2x_1 + 3x_2 + x_3 + x_4 &= 0 \\ x_1 + x_2 + 3x_3 - x_4 &= 0 \end{aligned}$$

(a) Before doing any arithmetic, explain why this system must have a solution other than $x_1 = x_2 = x_3 = x_4 = 0$.

(b) Solve it completely by Gauss–Jordan. Give both ranks, the leading and free variables, and the general solution in vector form. Check your answer at a non-zero value of the parameter.

(c) A classmate says: "Four unknowns and only three equations — so it has infinitely many solutions. That is a theorem; no work needed." Is the conclusion right? Is the reasoning?