

Linear Algebra — Class Exercise 4

Rank, Free Variables and the General Solution

Complete Solutions with Statements

Problem 1

Each matrix below is the reduced row echelon form of the augmented matrix of a system — no elimination is left to do. For each one state $\text{rank}(A)$, state $\text{rank}[A \mid b]$, say how many solutions the system has, and name the leading and the free variables. Where the solution set is infinite, write the general solution in vector form and say how many parameters the rank criterion predicted.

$$N_1 = \begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix}, \quad N_2 = \begin{bmatrix} 1 & 0 & 5 & 2 \\ 0 & 1 & -3 & 7 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$N_3 = \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad N_4 = \begin{bmatrix} 0 & 1 & 0 & -3 & 2 \\ 0 & 0 & 1 & 2 & -4 \end{bmatrix}.$$

Hint. Count the leading 1s first, then ask whether any of them stands in the last column. Both ranks come off the same matrix.

Solution:

Recall that for an augmented matrix $[A \mid b]$:

- The last column represents the constant vector b , while the preceding columns represent the coefficient matrix A corresponding to unknowns $x_1, x_2, \dots, x_n$.
- $\text{rank}(A)$ equals the number of leading 1s (pivots) in the coefficient columns.
- $\text{rank}[A \mid b]$ equals the total number of leading 1s in the entire augmented matrix.
- **Consistency criterion:**
 - If $\text{rank}(A) < \text{rank}[A \mid b]$, a leading 1 occurs in the constant column (an inconsistent equation $0 = 1$), so the system has **no solution** (0 solutions).
 - If $\text{rank}(A) = \text{rank}[A \mid b] = n$ (number of unknowns), the system has **exactly one solution** (unique solution).
 - If $\text{rank}(A) = \text{rank}[A \mid b] = r < n$, the system has **infinitely many solutions**, with $n - r$ free variables (parameters).

Analysis of Matrix N_1

$$[A \mid b] = \left[\begin{array}{cccc|c} 1 & -2 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 4 \end{array} \right]$$

- **Dimensions and unknowns:** Size is 3×5 . The coefficient matrix A has 4 columns, so there are $n = 4$ unknowns: x_1, x_2, x_3, x_4 .

- **Leading 1s (Pivots):**
 - Row 1: column 1 (x_1).
 - Row 2: column 3 (x_3).
 - Row 3: column 4 (x_4).
 - There is no leading 1 in the constant column (column 5).
- **Ranks:** $\text{rank}(A) = 3$, $\text{rank}[A \mid b] = 3$.
- **Number of solutions:** Consistent with $\text{rank}(A) = 3 < 4 = n \implies$ **infinitely many solutions**.
- **Variables:**
 - **Leading variables:** x_1, x_3, x_4 .
 - **Free variable:** x_2 .
- **Parameters predicted:** $n - \text{rank}(A) = 4 - 3 = 1$ parameter.
- **General solution in vector form:**

$$x_1 - 2x_2 = 3 \implies x_1 = 3 + 2x_2, \quad x_3 = -1, \quad x_4 = 4.$$

Setting free variable $x_2 = t \in \mathbb{R}$:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 + 2t \\ t \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ -1 \\ 4 \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad t \in \mathbb{R}.$$

Analysis of Matrix N_2

$$[A \mid b] = \left[\begin{array}{ccc|c} 1 & 0 & 5 & 2 \\ 0 & 1 & -3 & 7 \\ 0 & 0 & 0 & 1 \end{array} \right]$$

- **Dimensions and unknowns:** Size is 3×4 . The coefficient matrix A has 3 columns, so $n = 3$ unknowns: x_1, x_2, x_3 .
- **Leading 1s (Pivots):**
 - Row 1: column 1 (x_1).
 - Row 2: column 2 (x_2).
 - Row 3: column 4 (constant column).
- **Ranks:** $\text{rank}(A) = 2$, $\text{rank}[A \mid b] = 3$.
- **Number of solutions:** Since $\text{rank}(A) < \text{rank}[A \mid b]$ (row 3 states $0 = 1$), the system is **inconsistent: 0 solutions** (no solution).
- **Variables:** In coefficient matrix A , leading variables are x_1, x_2 and free variable is x_3 . Because the system is inconsistent, the solution set is empty ($\emptyset$).

Analysis of Matrix N_3

$$[A \mid b] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

- **Dimensions and unknowns:** Size is 4×4 . The coefficient matrix A has 3 columns, so $n = 3$ unknowns: x_1, x_2, x_3 .
- **Leading 1s (Pivots):** Row 1: col 1 (x_1); Row 2: col 2 (x_2); Row 3: col 3 (x_3). Row 4 is all zeros.
- **Ranks:** $\text{rank}(A) = 3$, $\text{rank}[A \mid b] = 3$.
- **Number of solutions:** Consistent with $\text{rank}(A) = \text{rank}[A \mid b] = n = 3 \implies$ **exactly 1 solution** (unique).
- **Variables:** Leading: x_1, x_2, x_3 . Free: **none**.
- **Parameters predicted:** $n - \text{rank}(A) = 3 - 3 = 0$ parameters.
- **Unique solution:** $\mathbf{x} = \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}$.

Analysis of Matrix N_4

$$[A \mid b] = \left[\begin{array}{cccc|c} 0 & 1 & 0 & -3 & 2 \\ 0 & 0 & 1 & 2 & -4 \end{array} \right]$$

- **Dimensions and unknowns:** Size is 2×5 . The coefficient matrix A has 4 columns, so $n = 4$ unknowns: x_1, x_2, x_3, x_4 .
- **Leading 1s (Pivots):** Row 1: column 2 (x_2); Row 2: column 3 (x_3). Neither is in the constant column.
- **Ranks:** $\text{rank}(A) = 2$, $\text{rank}[A \mid b] = 2$.
- **Number of solutions:** Consistent with $\text{rank}(A) = 2 < 4 = n \implies$ **infinitely many solutions**.
- **Variables:** Leading: x_2, x_3 . Free: x_1, x_4 .
- **Parameters predicted:** $n - \text{rank}(A) = 4 - 2 = 2$ parameters.
- **General solution in vector form:**

$$x_2 = 2 + 3x_4, \quad x_3 = -4 - 2x_4.$$

Setting $x_1 = s$ and $x_4 = t$ ($s, t \in \mathbb{R}$):

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ -4 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 3 \\ -2 \\ 1 \end{bmatrix}, \quad s, t \in \mathbb{R}.$$

Problem 2

Let c be a number, and consider

$$\begin{aligned}x_1 + 2x_2 - x_3 + 3x_4 &= 2 \\2x_1 + 4x_2 - x_3 + 8x_4 &= 7 \\3x_1 + 6x_2 - 4x_3 + 7x_4 &= c\end{aligned}$$

Reduce the augmented matrix by Gauss–Jordan and decide for which c the system is consistent, giving both ranks in each case. For the value of c that works, say which variables lead and which are free and write the general solution in vector form, checking it in the original system at a choice of the parameters that is not all zeros. Is there any c at all for which the system has exactly one solution?

Hint. The letter sits in the constant column and the row operations ignore the bar, so carry c along untouched — the coefficients decide the arithmetic and c only comes to matter at the last row.

Solution:**Step 1: Set up Augmented Matrix and Forward Elimination**

$$[A \mid b] = \left[\begin{array}{cccc|c} 1 & 2 & -1 & 3 & 2 \\ 2 & 4 & -1 & 8 & 7 \\ 3 & 6 & -4 & 7 & c \end{array} \right]$$

Perform row operations:

- $R_2 \leftarrow R_2 - 2R_1 \implies [0 \ 0 \ 1 \ 2 \mid 3]$
- $R_3 \leftarrow R_3 - 3R_1 \implies [0 \ 0 \ -1 \ -2 \mid c - 6]$

Matrix after clearing column 1 below the pivot:

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 3 & 2 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & -1 & -2 & c - 6 \end{array} \right] \xrightarrow{R_3 \leftarrow R_3 + R_2} \left[\begin{array}{cccc|c} 1 & 2 & -1 & 3 & 2 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & c - 3 \end{array} \right]$$

Step 2: Backward Elimination to RREF Eliminate above the pivot in column 3:

$$R_1 \leftarrow R_1 + R_2 \implies \left[\begin{array}{cccc|c} 1 & 2 & 0 & 5 & 5 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & c - 3 \end{array} \right]$$

Step 3: Consistency Analysis and Ranks

1. **Case $c \neq 3$:** Row 3 gives $[0 \ 0 \ 0 \ 0 \mid c - 3]$ with $c - 3 \neq 0$, creating a pivot in the constant column.

$$\text{rank}(A) = 2, \quad \text{rank}[A \mid b] = 3.$$

Since $\text{rank}(A) < \text{rank}[A \mid b]$, the system is **inconsistent (0 solutions)**.

2. **Case $c = 3$:** Row 3 becomes all zeros $[0 \ 0 \ 0 \ 0 \mid 0]$.

$$\text{rank}(A) = 2, \quad \text{rank}[A \mid b] = 2.$$

Since $\text{rank}(A) = \text{rank}[A \mid b] = 2 < 4 = n$, the system is **consistent with infinitely many solutions**.

Thus, the system is consistent **if and only if** $c = 3$.

Step 4: Variables and General Solution for $c = 3$

- **Leading variables:** x_1, x_3 .
- **Free variables:** x_2, x_4 .
- **Parameters predicted:** $n - \text{rank}(A) = 4 - 2 = 2$ parameters.

From RREF:

$$x_1 = 5 - 2x_2 - 5x_4, \quad x_3 = 3 - 2x_4.$$

Setting $x_2 = s$ and $x_4 = t$ ($s, t \in \mathbb{R}$):

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 3 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -5 \\ 0 \\ -2 \\ 1 \end{bmatrix}, \quad s, t \in \mathbb{R}.$$

Step 5: Verification in Original System Let $s = 1, t = 1$, yielding $\mathbf{x} = [-2, 1, 1, 1]^T$. Substituting into the original equations with $c = 3$:

- Eq 1: $(-2) + 2(1) - (1) + 3(1) = 2$ ✓
- Eq 2: $2(-2) + 4(1) - (1) + 8(1) = 7$ ✓
- Eq 3: $3(-2) + 6(1) - 4(1) + 7(1) = 3 = c$ ✓

Step 6: Can the System Have Exactly One Solution? No. A unique solution requires $\text{rank}(A) = n = 4$. But A has only 3 rows, and after row reduction $\text{rank}(A) = 2 \leq 3 < 4$ regardless of c (since c sits in the constant column and does not affect the row echelon structure of A). Therefore, the system can never have a unique solution for any value of c .

Problem 3

Consider the homogeneous system

$$x_1 + 2x_2 - x_3 + x_4 = 0$$

$$2x_1 + 3x_2 + x_3 + x_4 = 0$$

$$x_1 + x_2 + 3x_3 - x_4 = 0$$

- Before doing any arithmetic, explain why this system must have a solution other than $x_1 = x_2 = x_3 = x_4 = 0$.
- Solve it completely by Gauss–Jordan. Give both ranks, the leading and free variables, and the general solution in vector form. Check your answer at a non-zero value of the parameter.
- A classmate says: “Four unknowns and only three equations — so it has infinitely many solutions. That is a theorem; no work needed.” Is the conclusion right? Is the reasoning?

Solution:

(a) Explanation Prior to Arithmetic Every homogeneous system $A\mathbf{x} = \mathbf{0}$ is guaranteed to be consistent because it always has the trivial solution $\mathbf{x} = \mathbf{0}$. Here $m = 3$ equations and $n = 4$ unknowns. Since $\text{rank}(A) \leq m = 3$, the number of free variables is:

$$\text{free variables} = n - \text{rank}(A) \geq 4 - 3 = 1.$$

Because the system is consistent and has at least one free variable, it must have **infinitely many solutions**. Non-trivial solutions necessarily exist.

(b) Complete Solution by Gauss–Jordan Elimination Set up $[A \mid \mathbf{0}]$:

$$\begin{aligned} & \left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 0 \\ 2 & 3 & 1 & 1 & 0 \\ 1 & 1 & 3 & -1 & 0 \end{array} \right] \xrightarrow{\substack{R_2 \leftarrow R_2 - 2R_1 \\ R_3 \leftarrow R_3 - R_1}} \left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 0 \\ 0 & -1 & 3 & -1 & 0 \\ 0 & -1 & 4 & -2 & 0 \end{array} \right] \\ & \xrightarrow{R_2 \leftarrow -R_2} \left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 0 \\ 0 & 1 & -3 & 1 & 0 \\ 0 & -1 & 4 & -2 & 0 \end{array} \right] \xrightarrow{R_3 \leftarrow R_3 + R_2} \left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 0 \\ 0 & 1 & -3 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array} \right] \end{aligned}$$

Backward elimination:

$$\xrightarrow{\substack{R_2 \leftarrow R_2 + 3R_3 \\ R_1 \leftarrow R_1 + R_3}} \left[\begin{array}{cccc|c} 1 & 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array} \right] \xrightarrow{R_1 \leftarrow R_1 - 2R_2} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 4 & 0 \\ 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array} \right]$$

- **Ranks:** $\text{rank}(A) = 3$, $\text{rank}[A \mid \mathbf{0}] = 3$.
- **Leading variables:** x_1, x_2, x_3 .
- **Free variable:** x_4 .
- **Parameters predicted:** $n - \text{rank}(A) = 4 - 3 = 1$ parameter.
- **General solution:** $x_1 = -4x_4$, $x_2 = 2x_4$, $x_3 = x_4$. Setting $x_4 = t \in \mathbb{R}$:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = t \begin{bmatrix} -4 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \quad t \in \mathbb{R}.$$

Check at $t = 1$: $\mathbf{x} = [-4, 2, 1, 1]^T$:

- $(-4) + 2(2) - (1) + (1) = 0$ ✓
- $2(-4) + 3(2) + (1) + (1) = 0$ ✓
- $(-4) + (2) + 3(1) - (1) = 0$ ✓

(c) Critique of Classmate's Statement

- **Conclusion is right:** The system does possess infinitely many solutions.
- **Reasoning is flawed:** The theorem “ $n > m$ implies infinitely many solutions” applies **strictly to homogeneous systems** (or systems already established to be consistent). For general non-homogeneous systems, more unknowns than equations does not prevent contradictions (0 solutions). For example, $x_1 + x_2 + x_3 + x_4 = 1$ and $x_1 + x_2 + x_3 + x_4 = 2$ has 4 unknowns and 2 equations but no solutions. Because the classmate omitted the essential hypothesis that the system is homogeneous, the argument is logically incomplete.