Linear Algebra — Class Exercise 5

Parameters and Homogeneous Systems

1. Let k be a number, and consider

$$\begin{aligned} x + 2y + z &= 1 \\ 2x + 3y + 4z &= 3 \\ x + 2y + k^2 z &= k + 2 \end{aligned}$$

   (a) Reduce the augmented matrix, and determine for which values of k the system has no solution, exactly one solution, and infinitely many. Give the solution explicitly in each case where one exists.

   (b) State your answer in a single sentence of plain English.

   (c) Explain what goes wrong if, immediately after reaching the bottom row, you divide it through by its coefficient instead of splitting into cases.

   (d) Your two critical values behave differently. Describe, in terms of the three planes, what is different about the third plane in each case.

   Hint. Only interchanges and row replacements are safe here. Two replacements suffice, and neither of them divides by anything.

2. Let h and k be numbers, and consider

$$\begin{aligned} 2x + 3y &= 5 \\ 4x + hy &= k \end{aligned}$$

   (a) Discuss the system for *all* values of h and k: give the conditions under which it has exactly one solution, infinitely many, or none. Where the solution is unique, give x and y in terms of h and k.

   (b) Describe each of your three cases geometrically, as a statement about two lines in the plane.

   (c) Now fix $h = 6$, so that the system depends on k alone. Show that *no* value of k gives a unique solution, and explain why using $\text{rank}(A)$ rather than by inspecting the equations. When there is a solution, how many parameters does it need, and why could you have said so before solving?

   Hint. One row replacement reduces the whole problem to a single row. The two letters do not play the same role: one of them lives in the coefficient matrix and one in the constant column.

3. Let k be a number, and consider the homogeneous system

$$\begin{aligned} kx + y + z &= 0 \\ x + ky + z &= 0 \\ x + y + kz &= 0 \end{aligned}$$

(a) Reduce the system — without ever dividing by an expression containing k — and find every value of k for which it has a solution other than $x = y = z = 0$.

(b) For each such value, give the complete solution set in vector form, and describe it geometrically. The two critical values do *not* give solution sets of the same kind; say what the difference is and how the rank accounts for it.

(c) A classmate reports that for one value of k the system has no solution. Without doing any computation, explain how you know they are wrong.

(d) Delete the third equation, leaving two homogeneous equations in three unknowns. Explain why the resulting system has a non-trivial solution for *every* value of k, with no calculation at all.

Hint. Interchange rows first so that the pivot is the constant 1 rather than k. Later on you will need a replacement of the form $R_3 \to R_3 - (1 + k)R_2$ — multiplying the *other* row by an expression in k is a replacement and is perfectly safe; it is *dividing* that is forbidden.