

Linear Algebra — Class Exercise 5

Parameters and Homogeneous Systems

1. Let k be a number, and consider

$$\begin{aligned}x + 2y + z &= 1 \\2x + 3y + 4z &= 3 \\x + 2y + k^2z &= k + 2\end{aligned}$$

- (a) Reduce the augmented matrix, and determine for which values of k the system has no solution, exactly one solution, and infinitely many. Give the solution explicitly in each case where one exists.
- (b) State your answer in a single sentence of plain English.
- (c) Explain what goes wrong if, immediately after reaching the bottom row, you divide it through by its coefficient instead of splitting into cases.
- (d) Your two critical values behave differently. Describe, in terms of the three planes, what is different about the third plane in each case.

Hint. Only interchanges and row replacements are safe here. Two replacements suffice, and neither of them divides by anything.

Solution. (a) Reduce. The $(1, 1)$ entry is already a 1:

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 2 & 3 & 4 & 3 \\ 1 & 2 & k^2 & k+2 \end{array} \right] \xrightarrow{\substack{R_2 \rightarrow R_2 - 2R_1 \\ R_3 \rightarrow R_3 - R_1}} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & -1 & 2 & 1 \\ 0 & 0 & k^2 - 1 & k+1 \end{array} \right]$$

Nothing has been divided by, so this single reduction is valid for *every* k at once. Columns 1 and 2 carry pivots whatever happens, so the whole question is the bottom row,

$$(k^2 - 1)z = k + 1.$$

Critical values. Set the pivot expression to zero: $k^2 - 1 = 0$, so $k = 1$ or $k = -1$. Everything else is generic.

Case $k \neq 1$ and $k \neq -1$. The coefficient $k^2 - 1$ is non-zero, so column 3 gets a pivot and $\text{rank}(A) = \text{rank}[A \mid \mathbf{b}] = 3 = n$: **exactly one solution**. Only now is it legal to divide:

$$z = \frac{k+1}{k^2-1} = \frac{k+1}{(k-1)(k+1)} = \frac{1}{k-1}, \quad y = 2z - 1 = \frac{3-k}{k-1}, \quad x = 1 - 2y - z = \frac{3k-8}{k-1}.$$

(Row 2 reads $-y + 2z = 1$, so $y = 2z - 1$; row 1 then gives $x = 1 - 2y - z$.) Check at $k = 0$, where the system is $x + 2y + z = 1$, $2x + 3y + 4z = 3$, $x + 2y = 2$ and the formulas give $(x, y, z) = (8, -3, -1)$: $8 - 6 - 1 = 1$ ✓, $16 - 9 - 4 = 3$ ✓, $8 - 6 = 2$ ✓.

Case $k = 1$. The bottom row reads $0z = 2$, that is $0 = 2$: a contradiction. So $\text{rank}(A) = 2 < 3 = \text{rank}[A \mid \mathbf{b}]$ and there is **no solution**. Directly in the equations, the third one has become $x + 2y + z = 3$ while the first says $x + 2y + z = 1$.

Case $k = -1$. The bottom row reads $0z = 0$, that is $0 = 0$. So $\text{rank}(A) = \text{rank}[A \mid \mathbf{b}] = 2 < 3 = n$: **infinitely many solutions**, with $n - \text{rank}(A) = 1$ parameter. The third equation has become a copy of the first, and what is left is $x + 2y + z = 1$ and $-y + 2z = 1$. Put $z = t$; then $y = 2t - 1$ and $x = 1 - 2(2t - 1) - t = 3 - 5t$, so

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -5 \\ 2 \\ 1 \end{bmatrix}, \quad t \in \mathbb{R}.$$

Check at $t = 1$, i.e. $(-2, 1, 1)$, in the original system with $k = -1$: $-2 + 2 + 1 = 1 \checkmark$, $-4 + 3 + 4 = 3 \checkmark$, $-2 + 2 + 1 = 1 = k + 2 \checkmark$.

(b) In words. *The system has a unique solution for every value of k except 1 and -1 ; at $k = -1$ it has infinitely many solutions, forming a line; at $k = 1$ it has none.*

(c) Dividing too early destroys two of the three cases. Writing

$$z = \frac{k+1}{(k-1)(k+1)} = \frac{1}{k-1}$$

does two illegal things at once. The division by $k^2 - 1$ assumes $k \neq \pm 1$, and the cancellation of $k+1$ assumes $k \neq -1$ — and those are exactly the values the question is about. The result would be the claim “a unique solution for every $k \neq 1$ ”, which is wrong at $k = -1$ and gives no information about what happens at $k = 1$ beyond “the formula fails”.

The tell-tale sign: the pivot expression was a *quadratic*, so there were two critical values to account for, and an answer naming only one of them has lost a case. The correct order is always: find the roots first, split into cases, and divide only inside the case that has already excluded the roots.

(d) The third plane moves; the other two stand still. Equations 1 and 2 do not involve k , so they are two fixed, non-parallel planes meeting in a fixed line ℓ . All k does is slide and tilt the third plane, and the answer records how that plane meets ℓ .

At $k = -1$ the third plane has landed exactly on the first: only two distinct planes remain, and their intersection is the whole line ℓ — infinitely many solutions. At $k = 1$ the third plane has the same normal as the first, $(1, 2, 1)$, but a different constant, so it is *parallel* to the first and distinct from it: the three have no common point at all. In the generic case the third plane cuts across ℓ at a single point.

Coincident versus parallel is the whole difference, and it is invisible in the coefficient matrix alone — both cases kill the same pivot. Which of the two happens is decided by the constant column, which is why the critical values must be checked one at a time.

2. Let h and k be numbers, and consider

$$\begin{aligned} 2x + 3y &= 5 \\ 4x + hy &= k \end{aligned}$$

- (a) Discuss the system for *all* values of h and k : give the conditions under which it has exactly one solution, infinitely many, or none. Where the solution is unique, give x and y in terms of h and k .
- (b) Describe each of your three cases geometrically, as a statement about two lines in the plane.
- (c) Now fix $h = 6$, so that the system depends on k alone. Show that *no* value of k gives a unique solution, and explain why using $\text{rank}(A)$ rather than by inspecting the equations. When there is a solution, how many parameters does it need, and why could you have said so before solving?

Hint. One row replacement reduces the whole problem to a single row. The two letters do not play the same role: one of them lives in the coefficient matrix and one in the constant column.

Solution. (a) One replacement, $R_2 \rightarrow R_2 - 2R_1$:

$$\left[\begin{array}{cc|c} 2 & 3 & 5 \\ 4 & h & k \end{array} \right] \xrightarrow{R_2 \rightarrow R_2 - 2R_1} \left[\begin{array}{cc|c} 2 & 3 & 5 \\ 0 & h-6 & k-10 \end{array} \right]$$

The entire answer is in the bottom row, $(h-6)y = k-10$.

$h \neq 6$, *any* k : the coefficient is non-zero, so column 2 gets a pivot and $\text{rank}(A) = \text{rank}[A | \mathbf{b}] = 2 = n$ — **exactly one solution**. Dividing is legal now:

$$y = \frac{k-10}{h-6}, \quad x = \frac{5-3y}{2} = \frac{5(h-6) - 3(k-10)}{2(h-6)} = \frac{5h-3k}{2(h-6)}.$$

Check at $h = 4$, $k = 0$, where the system is $2x + 3y = 5$, $4x + 4y = 0$: the formulas give $y = \frac{-10}{-2} = 5$ and $x = \frac{20-0}{-4} = -5$, and indeed $-10 + 15 = 5$ ✓, $-20 + 20 = 0$ ✓.

$h = 6$ and $k = 10$: the bottom row is $0 = 0$, so $\text{rank}(A) = \text{rank}[A | \mathbf{b}] = 1 < 2 = n$ — **infinitely many solutions**, one parameter. With $y = t$, $x = \frac{1}{2}(5 - 3t)$.

$h = 6$ and $k \neq 10$: the bottom row is $0 = k - 10 \neq 0$, so $\text{rank}(A) = 1 < 2 = \text{rank}[A | \mathbf{b}]$ — **no solution**.

Notice how unequal the two letters are. h alone decides whether there is a critical case at all; k says nothing until $h = 6$ has flattened the left-hand side. That is because h lives in the coefficient matrix and so controls $\text{rank}(A)$ — the *how many* question — while k lives in the constant column and can only control whether $\text{rank}[A | \mathbf{b}]$ exceeds $\text{rank}(A)$ — the *are there any* question, which is in doubt only once a degenerate row has appeared.

(b) Two lines in the plane. Each equation is a line. $h \neq 6$: the lines have different slopes and cross at one point. $h = 6$, $k = 10$: the second equation is exactly twice the first, so the two lines coincide, and every point of that single line is a solution. $h = 6$, $k \neq 10$: the left-hand sides are still in the ratio 1 : 2 but the right-hand sides are not, so the lines are parallel and distinct — they never meet.

(c) With $h = 6$ the system is $2x + 3y = 5$, $4x + 6y = k$. Now

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix},$$

and k does not appear in it. Row 2 is twice row 1, so one replacement leaves a zero row and $\text{rank}(A) = 1$, *whatever k is*. Uniqueness requires $\text{rank}(A) = n = 2$, which is now unreachable. So no value of k gives exactly one solution — the answer is either none or infinitely many, and k only chooses between those two.

When there is a solution (at $k = 10$) it needs $n - \text{rank}(A) = 2 - 1 = 1$ parameter. That count could have been stated before solving, because it depends only on $\text{rank}(A)$ and n , and $\text{rank}(A)$ was fixed by the coefficient matrix alone.

This is the general rule about where a letter sits. A letter in the coefficient matrix can change the number of parameters and can create or destroy uniqueness. A letter in the constant column can do neither: it can only decide between “no solution” and “infinitely many” if $\text{rank}(A) < n$, or leave a unique solution unique if $\text{rank}(A) = n$.

3. Let k be a number, and consider the homogeneous system

$$\begin{aligned} kx + y + z &= 0 \\ x + ky + z &= 0 \\ x + y + kz &= 0 \end{aligned}$$

- Reduce the system — without ever dividing by an expression containing k — and find every value of k for which it has a solution other than $x = y = z = 0$.
- For each such value, give the complete solution set in vector form, and describe it geometrically. The two critical values do *not* give solution sets of the same kind; say what the difference is and how the rank accounts for it.
- A classmate reports that for one value of k the system has no solution. Without doing any computation, explain how you know they are wrong.
- Delete the third equation, leaving two homogeneous equations in three unknowns. Explain why the resulting system has a non-trivial solution for *every* value of k , with no calculation at all.

Hint. Interchange rows first so that the pivot is the constant 1 rather than k . Later on you will need a replacement of the form $R_3 \rightarrow R_3 - (1+k)R_2$ — multiplying the *other* row by an expression in k is a replacement and is perfectly safe; it is *dividing* that is forbidden.

Solution. (a) Reduce. Interchange $R_1 \leftrightarrow R_2$ so that the first pivot is a plain 1:

$$\left[\begin{array}{ccc|c} k & 1 & 1 & 0 \\ 1 & k & 1 & 0 \\ 1 & 1 & k & 0 \end{array} \right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & k & 1 & 0 \\ k & 1 & 1 & 0 \\ 1 & 1 & k & 0 \end{array} \right] \xrightarrow{\substack{R_2 \rightarrow R_2 - kR_1 \\ R_3 \rightarrow R_3 - R_1}} \left[\begin{array}{ccc|c} 1 & k & 1 & 0 \\ 0 & 1 - k^2 & 1 - k & 0 \\ 0 & 1 - k & k - 1 & 0 \end{array} \right]$$

Interchange again, to put the simpler row second, and then clear it:

$$\xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & k & 1 & 0 \\ 0 & 1 - k & k - 1 & 0 \\ 0 & 1 - k^2 & 1 - k & 0 \end{array} \right] \xrightarrow{R_3 \rightarrow R_3 - (1+k)R_2} \left[\begin{array}{ccc|c} 1 & k & 1 & 0 \\ 0 & 1 - k & k - 1 & 0 \\ 0 & 0 & (1 - k)(2 + k) & 0 \end{array} \right]$$

using $1 - k^2 = (1 + k)(1 - k)$ for the middle entry, and $(1 - k) - (1 + k)(k - 1) = (1 - k)[1 + (1 + k)] = (1 - k)(2 + k)$ for the last. Not one step divided by anything, so this holds for every k .

The system has only the trivial solution exactly when $\text{rank}(A) = 3 = n$, that is when both $1 - k$ and $(1 - k)(2 + k)$ are non-zero. So a **non-trivial solution occurs exactly when**

$$k = 1 \quad \text{or} \quad k = -2.$$

(b) The two critical values.

$k = 1$: rows 2 and 3 both vanish and the echelon form is $[1 \ 1 \ 1 \mid 0]$, so $\text{rank}(A) = 1$ and $n - \text{rank}(A) = 2$. All three original equations have become $x + y + z = 0$. With $y = s$ and $z = t$,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = s \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \quad s, t \in \mathbb{R} :$$

a **plane** through the origin — the three planes have collapsed onto one another. Check at $s = t = 1$, i.e. $(-2, 1, 1)$: every equation reads $-2 + 1 + 1 = 0$ ✓.

$k = -2$: row 2 becomes $[0 \ 3 \ -3 \mid 0]$ and row 3 vanishes, so $\text{rank}(A) = 2$ and $n - \text{rank}(A) = 1$. Row 2 gives $y = z$, and row 1, $x - 2y + z = 0$, gives $x = 2y - z = y$. With $z = t$,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad t \in \mathbb{R} :$$

a **line** through the origin — three distinct planes sharing a common line. Check at $t = 1$, i.e. $(1, 1, 1)$: $-2 + 1 + 1 = 0$ ✓, $1 - 2 + 1 = 0$ ✓, $1 + 1 - 2 = 0$ ✓.

Every other k : $\text{rank}(A) = 3 = n$, no free variables, only the trivial solution — three planes meeting at the origin alone.

The difference, and the rank. Both critical values give infinitely many solutions, but not the same amount of infinity. The number of parameters is $n - \text{rank}(A)$, so the dimension of the solution set is decided by how far the rank has fallen: at $k = -2$ it drops by one, from 3 to 2, and one parameter gives a line; at $k = 1$ it drops by two, from 3 to 1, and two parameters give a plane. A homogeneous system in three unknowns therefore has exactly three possible solution sets — the origin, a line through the origin, or a plane through the origin — according to $\text{rank}(A) = 3, 2$ or 1 .

A shortcut worth seeing. Adding all three equations gives $(k + 2)(x + y + z) = 0$. If $k \neq -2$ then $x + y + z = 0$; subtracting that from each original equation leaves $(k - 1)x = 0$, $(k - 1)y = 0$, $(k - 1)z = 0$, which forces the trivial solution unless $k = 1$. And if $k = -2$, subtracting the equations in pairs gives $x = y = z$. Same two critical values, no reduction required.

(c) “No solution” is impossible for a homogeneous system. Every constant term is zero, so $x = y = z = 0$ satisfies all three equations: the solution set is never empty. Structurally, row operations form combinations of rows and any combination of zeros is zero, so the augmented column stays $\mathbf{0}$ throughout, can never hold a leading 1, and therefore $\text{rank}(A) = \text{rank}[A \mid \mathbf{0}]$ always — the system is consistent by the rank criterion, whatever k is.

There are only *two* possible answers here, not three: the trivial solution alone, or infinitely many solutions. If a reduction of a homogeneous system ever produces a row $[0 \ 0 \ 0 \mid c]$ with $c \neq 0$, that is not a discovery but proof of an arithmetic slip.

(d) More unknowns than equations. Deleting an equation leaves $m = 2$ equations in $n = 3$ unknowns. Each leading 1 occupies a row of its own, so $\text{rank}(A) \leq m = 2 < 3 = n$. A homogeneous system is automatically consistent, so the criterion applies with its hypothesis already paid for: $\text{rank}(A) < n$ means at least $n - \text{rank}(A) \geq 1$ free variables, and setting one of them to a non-zero value produces a solution that is not all zeros.

No arithmetic is needed, and none of it depends on k . Note that the same counting says nothing at all for a *non*-homogeneous system with more unknowns than equations: there $\text{rank}(A) < n$ still rules out a unique solution, but the alternative may be “no solution” rather than “infinitely many”. Homogeneity is exactly what removes that possibility.