

Linear Algebra — Class Exercise 6

Matrix Algebra and Matrix Types

1. Let A be a 3×2 matrix, B a 2×3 matrix, and C a 3×2 matrix. For each of the following expressions, state whether it is defined. If it is defined, give the size of the resulting matrix; if it is undefined, explain why.

(a) $2A - 5C$

(b) $A + B$

(c) $A + B^T$

(d) $(3B - 4A^T)^T + 2C$

Hint. Addition requires identical dimensions; transposition flips $m \times n$ to $n \times m$.

Solution.

- (a) **Defined**, size 3×2 . Both A and C are 3×2 , so their scalar multiples $2A$ and $5C$ are 3×2 , and their difference has the same size 3×2 .
- (b) **Undefined**. A is 3×2 and B is 2×3 . Matrix addition is defined only when both matrices have identical dimensions.
- (c) **Defined**, size 3×2 . Since B is 2×3 , its transpose B^T is 3×2 , matching the size of A .
- (d) **Defined**, size 3×2 . A is 3×2 , so A^T is 2×3 , matching B . Thus $3B - 4A^T$ is 2×3 . Its transpose $(3B - 4A^T)^T$ is therefore 3×2 . Finally, adding $2C$ (which is 3×2) yields a 3×2 matrix.

2. Solve the following equation for the unknown 2×2 matrix X :

$$2(X - A^T) + B = X - 3I_2$$

where

$$A = \begin{bmatrix} 3 & -2 \\ 1 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 5 \\ 2 & 0 \end{bmatrix}, \quad I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Hint. Isolate X algebraically before substituting any matrices.

Solution. First isolate X using standard matrix algebra:

$$2X - 2A^T + B = X - 3I_2 \implies 2X - X = 2A^T - B - 3I_2 \implies X = 2A^T - B - 3I_2.$$

Now compute each term:

$$A = \begin{bmatrix} 3 & -2 \\ 1 & 4 \end{bmatrix} \implies A^T = \begin{bmatrix} 3 & 1 \\ -2 & 4 \end{bmatrix}, \quad 2A^T = \begin{bmatrix} 6 & 2 \\ -4 & 8 \end{bmatrix}.$$

Subtracting B :

$$2A^T - B = \begin{bmatrix} 6 - (-1) & 2 - 5 \\ -4 - 2 & 8 - 0 \end{bmatrix} = \begin{bmatrix} 7 & -3 \\ -6 & 8 \end{bmatrix}.$$

Subtracting $3I_2 = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$:

$$X = \begin{bmatrix} 7-3 & -3-0 \\ -6-0 & 8-3 \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ -6 & 5 \end{bmatrix}.$$

Check: $X - A^T = \begin{bmatrix} 4-3 & -3-1 \\ -6-(-2) & 5-4 \end{bmatrix} = \begin{bmatrix} 1 & -4 \\ -4 & 1 \end{bmatrix}$. Left side: $2(X - A^T) + B = \begin{bmatrix} 2 & -8 \\ -8 & 2 \end{bmatrix} + \begin{bmatrix} -1 & 5 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ -6 & 2 \end{bmatrix}$. Right side: $X - 3I_2 = \begin{bmatrix} 4-3 & -3-0 \\ -6-0 & 5-3 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ -6 & 2 \end{bmatrix} \checkmark$.

3. Decompose the matrix

$$M = \begin{bmatrix} 5 & -3 & 2 \\ 7 & 1 & -4 \\ -6 & 2 & 0 \end{bmatrix}$$

into the sum of a symmetric matrix S and a skew-symmetric matrix K , so that $M = S + K$. Write down S and K explicitly, and verify that $S^T = S$, $K^T = -K$, and $S + K = M$.

Hint. Use $S = \frac{1}{2}(M + M^T)$ and $K = \frac{1}{2}(M - M^T)$.

Solution. First write the transpose:

$$M^T = \begin{bmatrix} 5 & 7 & -6 \\ -3 & 1 & 2 \\ 2 & -4 & 0 \end{bmatrix}$$

The symmetric part is $S = \frac{1}{2}(M + M^T)$:

$$S = \frac{1}{2} \begin{bmatrix} 5+5 & -3+7 & 2+(-6) \\ 7+(-3) & 1+1 & -4+2 \\ -6+2 & 2+(-4) & 0+0 \end{bmatrix} = \begin{bmatrix} 5 & 2 & -2 \\ 2 & 1 & -1 \\ -2 & -1 & 0 \end{bmatrix}.$$

Check: $S^T = \begin{bmatrix} 5 & 2 & -2 \\ 2 & 1 & -1 \\ -2 & -1 & 0 \end{bmatrix} = S \checkmark$.

The skew-symmetric part is $K = \frac{1}{2}(M - M^T)$:

$$K = \frac{1}{2} \begin{bmatrix} 5-5 & -3-7 & 2-(-6) \\ 7-(-3) & 1-1 & -4-2 \\ -6-2 & 2-(-4) & 0-0 \end{bmatrix} = \begin{bmatrix} 0 & -5 & 4 \\ 5 & 0 & -3 \\ -4 & 3 & 0 \end{bmatrix}.$$

Check: $K^T = \begin{bmatrix} 0 & 5 & -4 \\ -5 & 0 & 3 \\ 4 & -3 & 0 \end{bmatrix} = -K \checkmark$. The main diagonal is all zeros, as skew-symmetry requires.

Adding them back:

$$S + K = \begin{bmatrix} 5+0 & 2-5 & -2+4 \\ 2+5 & 1+0 & -1-3 \\ -2-4 & -1+3 & 0+0 \end{bmatrix} = \begin{bmatrix} 5 & -3 & 2 \\ 7 & 1 & -4 \\ -6 & 2 & 0 \end{bmatrix} = M \checkmark.$$

4. Consider the upper triangular matrix parameterized by $k \in \mathbb{R}$:

$$A = \begin{bmatrix} 2k-1 & k-2 & 0 \\ 0 & k+1 & k^2-4 \\ 0 & 0 & 3k-3 \end{bmatrix}$$

- (a) For which value(s) of k is A a **diagonal** matrix?
- (b) For which value(s) of k is A a **symmetric** matrix? Explain why this condition is identical to part (a).
- (c) Can A ever be skew-symmetric? Justify your answer in one or two sentences.

Solution.

- (a) **Diagonal:** All off-diagonal entries must be zero. The entries below the diagonal are already zero. Above the diagonal, we require:

$$a_{12} = k - 2 = 0 \implies k = 2, \quad \text{and} \quad a_{23} = k^2 - 4 = 0 \implies k = 2 \text{ or } k = -2.$$

Both conditions hold simultaneously if and only if $k = 2$. (At $k = -2$, $a_{12} = -4 \neq 0$, so A is not diagonal.)

- (b) **Symmetric:** Symmetry requires $A^T = A$, meaning $a_{ij} = a_{ji}$ for all i, j . Because A is already upper triangular ($a_{21} = a_{31} = a_{32} = 0$), symmetry demands that the corresponding entries above the diagonal also be zero: $a_{12} = a_{21} = 0$ and $a_{23} = a_{32} = 0$. Thus, for an upper triangular matrix, being symmetric is *equivalent* to being diagonal. Hence A is symmetric if and only if $k = 2$.
- (c) **Skew-symmetry:** No, A can **never** be skew-symmetric. Skew-symmetry requires all diagonal entries to be zero. Here the diagonal entries are $2k - 1 = 0 \implies k = 1/2$, but $k + 1 = 0 \implies k = -1$, and $3k - 3 = 0 \implies k = 1$. Because k cannot simultaneously equal $1/2$, -1 , and 1 , the diagonal can never be entirely zero.