

Linear Algebra — Class Exercise 7

Matrix Multiplication and Powers

1. Consider the matrices

$$M = \begin{bmatrix} 1 & 2 \\ -1 & 3 \\ 2 & 0 \end{bmatrix}, \quad N = \begin{bmatrix} 3 & -1 \\ 2 & 4 \end{bmatrix}.$$

- (a) Compute the matrix product MN .
- (b) Is the reverse product NM defined? Explain why or why not.

2. Application to Economics: Multi-Plant Production and Factor Requirements.

A manufacturing firm, *Apex Electronics*, produces three consumer goods at two regional production plants: G_1 (smartwatches), G_2 (tablets), and G_3 (laptops). Production requires two scarce input factors: labor (measured in hours, L) and specialized semiconductor microchips (measured in units, M).

Given Data:

- Planned monthly output for Plant 1 (Ohio) and Plant 2 (Texas) is given by the 2×3 matrix Q (rows: plants; columns: goods G_1, G_2, G_3):

$$Q = \begin{bmatrix} 100 & 200 & 150 \\ 120 & 180 & 200 \end{bmatrix}.$$

- Technological input requirements per unit of good are given by the 3×2 matrix A (rows: goods; col 1: labor hours; col 2: microchips), and factor market prices (wage \$25/hour, microchip \$40/unit) are given by w :

$$A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \\ 5 & 4 \end{bmatrix}, \quad w = \begin{bmatrix} 25 \\ 40 \end{bmatrix}.$$

(a) Total Factor Demand Matrix:

- State the dimensions of the product $R = QA$.
- Compute the matrix product $R = QA$.
- Provide an economic interpretation for the entry R_{12} and the entry R_{21} .

(b) Unit Cost Vector:

- Compute the unit production cost vector $c = Aw$.
- State the dimension of c and interpret what each scalar entry represents.

(c) Total Factor Expenditure (Associative Property):

- Total input expenditure across both plants can be computed as either $E = Rw$ or $E = Qc$. Explain the economic intuition behind both computational paths.
- Compute E using both methods to verify that $(QA)w = Q(Aw)$.
- What is the total factor expenditure for Plant 1 and Plant 2, respectively?

3. Consider the matrices

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 \\ -1 & 3 \\ 0 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 3 & -1 \\ 4 & 2 \end{bmatrix}.$$

(a) For each expression, state whether it is defined. If defined, give its size; if undefined, explain why:
(i) $AB - 2C$ (ii) $BA + C$ (iii) BC^T .

(b) Compute the product AB , and then evaluate the matrix expression $AB - 2C$.

4. In ordinary real algebra, $(x + y)^2 = x^2 + 2xy + y^2$.

(a) Using the distributive property, expand $(A + B)^2 = (A + B)(A + B)$ for general $n \times n$ matrices A and B . What condition on A and B is required for $(A + B)^2 = A^2 + 2AB + B^2$ to hold?

(b) For $A = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 3 \\ 1 & 0 \end{bmatrix}$, compute AB and BA . Does $(A + B)^2 = A^2 + 2AB + B^2$ hold for this pair?

5. Consider the matrix $A = \begin{bmatrix} 3 & -2 \\ 4 & -3 \end{bmatrix}$.

(a) Compute A^2 .

(b) Using your result from part (a), find A^{50} and A^{75} without performing repeated multiplications.

(c) Expand the product $(A - I_2)(A + I_2)$ algebraically, evaluate it, and comment on what this reveals about the zero-factor property in matrix multiplication.

6. Let $A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 3 & 2 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$.

(a) Compute the product $A\mathbf{x}$ using the **column-combination view**, writing the calculation explicitly as a linear combination of the columns of A .

(b) Verify your answer by computing $A\mathbf{x}$ using the standard row-against-column dot product rule.

(c) Can $\mathbf{b}$ be written as a linear combination of the first two columns of A , namely $\mathbf{a}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\mathbf{a}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$? If so, find the weights c_1, c_2 such that $c_1\mathbf{a}_1 + c_2\mathbf{a}_2 = \mathbf{b}$.