

Linear Algebra — Class Exercise 7

Matrix Multiplication and Powers

1. Consider the matrices

$$M = \begin{bmatrix} 1 & 2 \\ -1 & 3 \\ 2 & 0 \end{bmatrix}, \quad N = \begin{bmatrix} 3 & -1 \\ 2 & 4 \end{bmatrix}.$$

- (a) Compute the matrix product MN .
 (b) Is the reverse product NM defined? Explain why or why not.

Solution.

- (a) M has size 3×2 and N has size 2×2 . The inner dimensions match $((3 \times \underline{2})(\underline{2} \times 2))$, so MN is a 3×2 matrix:

$$MN = \begin{bmatrix} 1(3) + 2(2) & 1(-1) + 2(4) \\ -1(3) + 3(2) & -1(-1) + 3(4) \\ 2(3) + 0(2) & 2(-1) + 0(4) \end{bmatrix} = \begin{bmatrix} 3 + 4 & -1 + 8 \\ -3 + 6 & 1 + 12 \\ 6 + 0 & -2 + 0 \end{bmatrix} = \begin{bmatrix} 7 & 7 \\ 3 & 13 \\ 6 & -2 \end{bmatrix}.$$

- (b) **Undefined.** N is 2×2 and M is 3×2 . For the product NM , the number of columns of N (which is 2) does not equal the number of rows of M (which is 3). Thus the inner dimensions do not match $((2 \times \underline{2})(\underline{3} \times 2))$.

2. Application to Economics: Multi-Plant Production and Factor Requirements.

A manufacturing firm, *Apex Electronics*, produces three consumer goods at two regional production plants: G_1 (smartwatches), G_2 (tablets), and G_3 (laptops). Production requires two scarce input factors: labor (measured in hours, L) and specialized semiconductor microchips (measured in units, M).

Given Data:

- Planned monthly output for Plant 1 (Ohio) and Plant 2 (Texas) is given by the 2×3 matrix Q (rows: plants; columns: goods G_1, G_2, G_3):

$$Q = \begin{bmatrix} 100 & 200 & 150 \\ 120 & 180 & 200 \end{bmatrix}.$$

- Technological input requirements per unit of good are given by the 3×2 matrix A (rows: goods; col 1: labor hours; col 2: microchips), and factor market prices (wage \$25/hour, microchip \$40/unit) are given by w :

$$A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \\ 5 & 4 \end{bmatrix}, \quad w = \begin{bmatrix} 25 \\ 40 \end{bmatrix}.$$

- (a) **Total Factor Demand Matrix:**

- State the dimensions of the product $R = QA$.
- Compute the matrix product $R = QA$.
- Provide an economic interpretation for the entry R_{12} and the entry R_{21} .

(b) **Unit Cost Vector:**

- i. Compute the unit production cost vector $c = Aw$.
- ii. State the dimension of c and interpret what each scalar entry represents.

(c) **Total Factor Expenditure (Associative Property):**

- i. Total input expenditure across both plants can be computed as either $E = Rw$ or $E = Qc$. Explain the economic intuition behind both computational paths.
- ii. Compute E using both methods to verify that $(QA)w = Q(Aw)$.
- iii. What is the total factor expenditure for Plant 1 and Plant 2, respectively?

Solution.

(a) **Factor Demands ($R = QA$):**

- i. Q is 2×3 and A is 3×2 . The product $R = QA$ has dimension $(2 \times 3)(3 \times 2) \rightarrow 2 \times 2$.
- ii. Matrix multiplication:

$$R = \begin{bmatrix} 100 & 200 & 150 \\ 120 & 180 & 200 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 3 & 2 \\ 5 & 4 \end{bmatrix}$$

$$R_{11} = 100(2) + 200(3) + 150(5) = 200 + 600 + 750 = 1,550,$$

$$R_{12} = 100(1) + 200(2) + 150(4) = 100 + 400 + 600 = 1,100,$$

$$R_{21} = 120(2) + 180(3) + 200(5) = 240 + 540 + 1,000 = 1,780,$$

$$R_{22} = 120(1) + 180(2) + 200(4) = 120 + 360 + 800 = 1,280.$$

$$R = \begin{bmatrix} 1,550 & 1,100 \\ 1,780 & 1,280 \end{bmatrix}.$$

- iii. $R_{12} = 1,100$: Plant 1 requires a total of 1,100 microchips to satisfy its monthly production schedule.
 $R_{21} = 1,780$: Plant 2 requires a total of 1,780 labor hours to satisfy its monthly production schedule.

(b) **Unit Cost Vector ($c = Aw$):**

- i. Computation:

$$c = \begin{bmatrix} 2 & 1 \\ 3 & 2 \\ 5 & 4 \end{bmatrix} \begin{bmatrix} 25 \\ 40 \end{bmatrix} = \begin{bmatrix} 2(25) + 1(40) \\ 3(25) + 2(40) \\ 5(25) + 4(40) \end{bmatrix} = \begin{bmatrix} 50 + 40 \\ 75 + 80 \\ 125 + 160 \end{bmatrix} = \begin{bmatrix} 90 \\ 155 \\ 285 \end{bmatrix}.$$

- ii. c has dimension 3×1 . Entry c_j represents the marginal/variable input cost to manufacture one unit of good G_j : \$90 per smartwatch (G_1), \$155 per tablet (G_2), and \$285 per laptop (G_3).

(c) **Total Factor Expenditure ($E = Rw = Qc$):**

- i. $E = Rw$ calculates total plant expenditure by multiplying total factor quantities consumed at each plant by input factor prices (Factor Market perspective). $E = Qc$ aggregates total expenditure by multiplying each plant's output quantities of each good by their respective unit production costs (Product Market perspective).

ii. Computing both paths:

$$Rw = \begin{bmatrix} 1,550 & 1,100 \\ 1,780 & 1,280 \end{bmatrix} \begin{bmatrix} 25 \\ 40 \end{bmatrix} = \begin{bmatrix} 1,550(25) + 1,100(40) \\ 1,780(25) + 1,280(40) \end{bmatrix} = \begin{bmatrix} 38,750 + 44,000 \\ 44,500 + 51,200 \end{bmatrix} = \begin{bmatrix} 82,750 \\ 95,700 \end{bmatrix}.$$

$$Qc = \begin{bmatrix} 100 & 200 & 150 \\ 120 & 180 & 200 \end{bmatrix} \begin{bmatrix} 90 \\ 155 \\ 285 \end{bmatrix} = \begin{bmatrix} 100(90) + 200(155) + 150(285) \\ 120(90) + 180(155) + 200(285) \end{bmatrix} = \begin{bmatrix} 82,750 \\ 95,700 \end{bmatrix}.$$

Both paths yield the identical expenditure vector $E = \begin{bmatrix} 82,750 \\ 95,700 \end{bmatrix}$, confirming the associative property $(QA)w = Q(Aw)$.

iii. The total factor expenditure is **\$82,750** for Plant 1 and **\$95,700** for Plant 2.

3. Consider the matrices

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 \\ -1 & 3 \\ 0 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 3 & -1 \\ 4 & 2 \end{bmatrix}.$$

- (a) For each expression, state whether it is defined. If defined, give its size; if undefined, explain why:
 (i) $AB - 2C$ (ii) $BA + C$ (iii) BC^T .
 (b) Compute the product AB , and then evaluate the matrix expression $AB - 2C$.

Solution.

- (a) • **(i) $AB - 2C$ is defined**, size 2×2 . A is 2×3 and B is 3×2 , so their product AB has size $(2 \times \underline{3})(\underline{3} \times 2) \rightarrow 2 \times 2$. Since C is 2×2 , the scalar multiple $2C$ is also 2×2 , matching the size of AB .
 • **(ii) $BA + C$ is undefined**. B is 3×2 and A is 2×3 , so BA is $(3 \times \underline{2})(\underline{2} \times 3) \rightarrow 3 \times 3$. But C is 2×2 . Matrix addition requires identical dimensions ($3 \times 3 \neq 2 \times 2$).
 • **(iii) BC^T is defined**, size 3×2 . B is 3×2 . Since C is 2×2 , its transpose C^T is also 2×2 . Thus BC^T has size $(3 \times \underline{2})(\underline{2} \times 2) \rightarrow 3 \times 2$.
 (b) We compute AB using the row-by-column rule:

$$AB = \begin{bmatrix} 1(2) + (-2)(-1) + 3(0) & 1(1) + (-2)(3) + 3(2) \\ 2(2) + 0(-1) + (-1)(0) & 2(1) + 0(3) + (-1)(2) \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ 4 & 0 \end{bmatrix}.$$

$$\text{Subtracting } 2C = 2 \begin{bmatrix} 3 & -1 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 6 & -2 \\ 8 & 4 \end{bmatrix}:$$

$$AB - 2C = \begin{bmatrix} 4 - 6 & 1 - (-2) \\ 4 - 8 & 0 - 4 \end{bmatrix} = \begin{bmatrix} -2 & 3 \\ -4 & -4 \end{bmatrix}.$$

4. In ordinary real algebra, $(x + y)^2 = x^2 + 2xy + y^2$.

- (a) Using the distributive property, expand $(A + B)^2 = (A + B)(A + B)$ for general $n \times n$ matrices A and B . What condition on A and B is required for $(A + B)^2 = A^2 + 2AB + B^2$ to hold?
 (b) For $A = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 3 \\ 1 & 0 \end{bmatrix}$, compute AB and BA . Does $(A + B)^2 = A^2 + 2AB + B^2$ hold for this pair?

Solution.

- (a) Expanding using distributivity (carefully preserving factor order):

$$(A + B)^2 = (A + B)(A + B) = A(A + B) + B(A + B) = A^2 + AB + BA + B^2.$$

In general, $AB \neq BA$. The expression equals $A^2 + 2AB + B^2$ if and only if $BA = AB$, that is, if and only if A and B **commute**.

- (b) Computing both products:

$$AB = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 3 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2(0) + 1(1) & 2(3) + 1(0) \\ 0(0) + 1(1) & 0(3) + 1(0) \end{bmatrix} = \begin{bmatrix} 1 & 6 \\ 1 & 0 \end{bmatrix},$$

$$BA = \begin{bmatrix} 0 & 3 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0(2) + 3(0) & 0(1) + 3(1) \\ 1(2) + 0(0) & 1(1) + 0(1) \end{bmatrix} = \begin{bmatrix} 0 & 3 \\ 2 & 1 \end{bmatrix}.$$

Since $AB \neq BA$, the matrices do not commute ($AB + BA \neq 2AB$). Therefore, $(A + B)^2 = A^2 + 2AB + B^2$ **does not hold** for this pair.

5. Consider the matrix $A = \begin{bmatrix} 3 & -2 \\ 4 & -3 \end{bmatrix}$.

- (a) Compute A^2 .
- (b) Using your result from part (a), find A^{50} and A^{75} without performing repeated multiplications.
- (c) Expand the product $(A - I_2)(A + I_2)$ algebraically, evaluate it, and comment on what this reveals about the zero-factor property in matrix multiplication.

Solution.

- (a) Computing
- A^2
- :

$$A^2 = \begin{bmatrix} 3 & -2 \\ 4 & -3 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ 4 & -3 \end{bmatrix} = \begin{bmatrix} 3(3) + (-2)(4) & 3(-2) + (-2)(-3) \\ 4(3) + (-3)(4) & 4(-2) + (-3)(-3) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2.$$

A matrix satisfying $A^2 = I$ is called an *involution*.

- (b) Using
- $A^2 = I_2$
- and the laws of exponents:

$$A^{50} = (A^2)^{25} = I_2^{25} = I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A^{75} = (A^2)^{37} A = I_2 A = A = \begin{bmatrix} 3 & -2 \\ 4 & -3 \end{bmatrix}.$$

In general, $A^k = I_2$ for any even integer $k \geq 1$, and $A^k = A$ for any odd integer $k \geq 1$.

- (c) Since any matrix commutes with the identity (
- $AI_2 = I_2A = A$
-), we have:

$$(A - I_2)(A + I_2) = A^2 + AI_2 - I_2A - I_2^2 = A^2 - I_2.$$

Because $A^2 = I_2$, this gives:

$$(A - I_2)(A + I_2) = I_2 - I_2 = O_{2 \times 2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Observation: Note that

$$A - I_2 = \begin{bmatrix} 2 & -2 \\ 4 & -4 \end{bmatrix} \neq O \quad \text{and} \quad A + I_2 = \begin{bmatrix} 4 & -2 \\ 4 & -2 \end{bmatrix} \neq O.$$

Neither factor is the zero matrix, yet their product is the zero matrix. This demonstrates that **the zero-factor property does not hold** for matrix multiplication: $MN = O$ does not imply $M = O$ or $N = O$. Matrices can be zero divisors.

6. Let $A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 3 & 2 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$.

- (a) Compute the product $A\mathbf{x}$ using the **column-combination view**, writing the calculation explicitly as a linear combination of the columns of A .
- (b) Verify your answer by computing $A\mathbf{x}$ using the standard row-against-column dot product rule.
- (c) Can $\mathbf{b}$ be written as a linear combination of the first two columns of A , namely $\mathbf{a}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\mathbf{a}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$? If so, find the weights c_1, c_2 such that $c_1\mathbf{a}_1 + c_2\mathbf{a}_2 = \mathbf{b}$.

Solution.

- (a) In the column-combination view, $A\mathbf{x}$ is the linear combination of the columns of A with weights from $\mathbf{x}$:

$$A\mathbf{x} = 3 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + (-2) \begin{bmatrix} 1 \\ 3 \end{bmatrix} + 4 \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 - 2 - 4 \\ 3 - 6 + 8 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \end{bmatrix}.$$

- (b) Using the row-against-column dot product rule:

$$A\mathbf{x} = \begin{bmatrix} 2(3) + 1(-2) + (-1)(4) \\ 1(3) + 3(-2) + 2(4) \end{bmatrix} = \begin{bmatrix} 6 - 2 - 4 \\ 3 - 6 + 8 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \end{bmatrix}.$$

This confirms the result from part (a) ✓.

- (c) We seek scalars c_1, c_2 such that:

$$c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 7 \\ 1 \end{bmatrix} \iff \begin{cases} 2c_1 + c_2 = 7 \\ c_1 + 3c_2 = 1 \end{cases}$$

From the second equation, $c_1 = 1 - 3c_2$. Substituting into the first equation:

$$2(1 - 3c_2) + c_2 = 7 \implies 2 - 5c_2 = 7 \implies -5c_2 = 5 \implies c_2 = -1.$$

Then $c_1 = 1 - 3(-1) = 4$.

Check: $4 \begin{bmatrix} 2 \\ 1 \end{bmatrix} - 1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 8 - 1 \\ 4 - 3 \end{bmatrix} = \begin{bmatrix} 7 \\ 1 \end{bmatrix} = \mathbf{b}$ ✓.

Therefore, $\mathbf{b}$ can indeed be written as a linear combination of $\mathbf{a}_1$ and $\mathbf{a}_2$, with weights $c_1 = 4$ and $c_2 = -1$.