

# Linear Algebra — Class Exercise 8

## Matrix Multiplication and Powers (Part 2)

### 1. Application to Survey Research: Fieldwork Budget of a Two-City Survey.

A public-opinion research institute is running a household survey in two cities, Northvale (City 1) and Southport (City 2). Interviews are completed in three modes:  $F$  (face-to-face),  $T$  (telephone), and  $O$  (online). Each completed interview uses two resources: interviewer time (in hours) and data-processing time for coding and cleaning the answers (in hours).

#### Given Data:

- The planned number of completed interviews is given by the  $2 \times 3$  matrix  $Q$  (rows: cities; columns: modes  $F, T, O$ ):

$$Q = \begin{bmatrix} 200 & 300 & 500 \\ 150 & 400 & 450 \end{bmatrix}.$$

- The time needed per completed interview is given by the  $3 \times 2$  matrix  $A$  (rows: modes  $F, T, O$ ; col 1: interviewer hours; col 2: data-processing hours), and the hourly pay rates (interviewers \$30/hour, data processors \$20/hour) are given by  $w$ :

$$A = \begin{bmatrix} 2 & 0.5 \\ 0.5 & 0.25 \\ 0 & 0.1 \end{bmatrix}, \quad w = \begin{bmatrix} 30 \\ 20 \end{bmatrix}.$$

#### (a) Staff Hours Matrix:

- State the dimensions of the product  $R = QA$ .
- Compute the matrix product  $R = QA$ .
- Interpret the entries  $R_{12}$  and  $R_{21}$  in the context of the survey. Why does the entry 0 in  $A$  make sense?

#### (b) Cost per Interview:

- Compute the vector  $c = Aw$ .
- State the dimension of  $c$  and interpret each entry. Which interview mode is cheapest?

#### (c) Fieldwork Budget (Associative Property):

- The fieldwork budget of each city can be computed as either  $E = Rw$  or  $E = Qc$ . Explain in words what each of the two computations does.
- Compute  $E$  both ways to verify that  $(QA)w = Q(Aw)$ .
- What is the fieldwork budget for City 1 and for City 2?

2. **Transposes and Symmetry.** Consider the matrices

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 \\ -1 & 0 \\ 1 & 4 \end{bmatrix}.$$

- (a) Give the sizes of  $AB$ ,  $(AB)^T$ ,  $B^T A^T$ , and  $A^T B^T$ . A student claims that  $(AB)^T = A^T B^T$ . Without computing any entries, explain why this cannot be true for these matrices, even though  $A^T B^T$  is defined.
- (b) Compute  $AB$  and  $B^T A^T$ , and check that  $(AB)^T = B^T A^T$ .
- (c) Compute  $AA^T$  and observe that it is symmetric. Then use the rule  $(XY)^T = Y^T X^T$  to prove that  $MM^T$  is symmetric for *every* matrix  $M$ .

3. **Powers at Work: Movements Between Employment and Unemployment.** A labour-market panel study follows the same group of young adults month after month. Each month, 90% of those who are employed stay employed and 10% become unemployed, while 40% of those who are unemployed find a job and 60% remain unemployed. With the states ordered as (Employed, Unemployed), this gives the **transition matrix**

$$P = \begin{bmatrix} 0.90 & 0.40 \\ 0.10 & 0.60 \end{bmatrix} \quad (\text{column } j = \text{state this month, row } i = \text{state next month}).$$

If  $\mathbf{x}_k$  holds the shares of the group that are employed and unemployed in month  $k$ , then  $\mathbf{x}_{k+1} = P\mathbf{x}_k$ .

- (a) Interpret the entry  $p_{21}$ . Explain why each column of  $P$  must add up to 1.
- (b) Compute  $P^2$ . Interpret the entry  $(P^2)_{12}$ , and explain why it is *not* equal to  $(p_{12})^2 = 0.16$ .
- (c) In month 0, 60% of the group are employed:  $\mathbf{x}_0 = [0.60 \ 0.40]^T$ . Compute  $\mathbf{x}_2$  in two ways, as  $P(P\mathbf{x}_0)$  and as  $P^2\mathbf{x}_0$ . Which property of matrix multiplication guarantees the same answer? If you had to forecast one year ahead for many different starting groups, which way would you choose?
- (d) Show that  $\mathbf{q} = [0.8 \ 0.2]^T$  satisfies  $P\mathbf{q} = \mathbf{q}$ , and deduce that  $P^k\mathbf{q} = \mathbf{q}$  for every  $k \geq 1$ . Interpret this result, and compare it with the shares you found in part (c).