

Linear Algebra — Class Exercise 8

Matrix Multiplication and Powers (Part 2)

1. Application to Survey Research: Fieldwork Budget of a Two-City Survey.

A public-opinion research institute is running a household survey in two cities, Northvale (City 1) and Southport (City 2). Interviews are completed in three modes: F (face-to-face), T (telephone), and O (online). Each completed interview uses two resources: interviewer time (in hours) and data-processing time for coding and cleaning the answers (in hours).

Given Data:

- The planned number of completed interviews is given by the 2×3 matrix Q (rows: cities; columns: modes F, T, O):

$$Q = \begin{bmatrix} 200 & 300 & 500 \\ 150 & 400 & 450 \end{bmatrix}.$$

- The time needed per completed interview is given by the 3×2 matrix A (rows: modes F, T, O ; col 1: interviewer hours; col 2: data-processing hours), and the hourly pay rates (interviewers \$30/hour, data processors \$20/hour) are given by w :

$$A = \begin{bmatrix} 2 & 0.5 \\ 0.5 & 0.25 \\ 0 & 0.1 \end{bmatrix}, \quad w = \begin{bmatrix} 30 \\ 20 \end{bmatrix}.$$

(a) Staff Hours Matrix:

- State the dimensions of the product $R = QA$.
- Compute the matrix product $R = QA$.
- Interpret the entries R_{12} and R_{21} in the context of the survey. Why does the entry 0 in A make sense?

(b) Cost per Interview:

- Compute the vector $c = Aw$.
- State the dimension of c and interpret each entry. Which interview mode is cheapest?

(c) Fieldwork Budget (Associative Property):

- The fieldwork budget of each city can be computed as either $E = Rw$ or $E = Qc$. Explain in words what each of the two computations does.
- Compute E both ways to verify that $(QA)w = Q(Aw)$.
- What is the fieldwork budget for City 1 and for City 2?

Solution.

(a) Staff Hours ($R = QA$):

- Q is 2×3 and A is 3×2 . The product $R = QA$ has dimension $(2 \times 3)(3 \times 2) \rightarrow 2 \times 2$ (rows: cities; columns: types of staff hours).

ii. Matrix multiplication:

$$R = \begin{bmatrix} 200 & 300 & 500 \\ 150 & 400 & 450 \end{bmatrix} \begin{bmatrix} 2 & 0.5 \\ 0.5 & 0.25 \\ 0 & 0.1 \end{bmatrix}$$

$$R_{11} = 200(2) + 300(0.5) + 500(0) = 400 + 150 + 0 = 550,$$

$$R_{12} = 200(0.5) + 300(0.25) + 500(0.1) = 100 + 75 + 50 = 225,$$

$$R_{21} = 150(2) + 400(0.5) + 450(0) = 300 + 200 + 0 = 500,$$

$$R_{22} = 150(0.5) + 400(0.25) + 450(0.1) = 75 + 100 + 45 = 220.$$

$$R = \begin{bmatrix} 550 & 225 \\ 500 & 220 \end{bmatrix}.$$

iii. $R_{12} = 225$: the survey in City 1 needs a total of 225 data-processing hours.

$R_{21} = 500$: the survey in City 2 needs a total of 500 interviewer hours.

The entry $A_{31} = 0$ says an online interview uses no interviewer time: respondents fill in the questionnaire themselves. Consequently, the 500 and 450 online interviews contribute nothing to the interviewer hours R_{11} and R_{21} .

(b) **Cost per Interview** ($c = Aw$):

i. Computation:

$$c = \begin{bmatrix} 2 & 0.5 \\ 0.5 & 0.25 \\ 0 & 0.1 \end{bmatrix} \begin{bmatrix} 30 \\ 20 \end{bmatrix} = \begin{bmatrix} 2(30) + 0.5(20) \\ 0.5(30) + 0.25(20) \\ 0(30) + 0.1(20) \end{bmatrix} = \begin{bmatrix} 60 + 10 \\ 15 + 5 \\ 0 + 2 \end{bmatrix} = \begin{bmatrix} 70 \\ 20 \\ 2 \end{bmatrix}.$$

ii. c has dimension 3×1 . Entry c_j is the staff cost of one completed interview in mode j : \$70 per face-to-face interview, \$20 per telephone interview, and \$2 per online interview. Online interviewing is by far the cheapest mode.

(c) **Fieldwork Budget** ($E = Rw = Qc$):

i. $E = Rw$ first adds up the total staff hours of each type needed in each city, and then pays for these hours at the hourly rates (“count the hours, then pay the staff”). $E = Qc$ first finds the cost of a single interview in each mode, and then multiplies by the number of interviews of each mode planned in each city (“price one interview, then count the interviews”).

ii. Computing both paths:

$$Rw = \begin{bmatrix} 550 & 225 \\ 500 & 220 \end{bmatrix} \begin{bmatrix} 30 \\ 20 \end{bmatrix} = \begin{bmatrix} 550(30) + 225(20) \\ 500(30) + 220(20) \end{bmatrix} = \begin{bmatrix} 16,500 + 4,500 \\ 15,000 + 4,400 \end{bmatrix} = \begin{bmatrix} 21,000 \\ 19,400 \end{bmatrix}.$$

$$Qc = \begin{bmatrix} 200 & 300 & 500 \\ 150 & 400 & 450 \end{bmatrix} \begin{bmatrix} 70 \\ 20 \\ 2 \end{bmatrix} = \begin{bmatrix} 14,000 + 6,000 + 1,000 \\ 10,500 + 8,000 + 900 \end{bmatrix} = \begin{bmatrix} 21,000 \\ 19,400 \end{bmatrix}.$$

Both paths give the same vector E , confirming the associative property $(QA)w = Q(Aw)$.

iii. The fieldwork budget is **\$21,000** for City 1 and **\$19,400** for City 2.

2. **Transposes and Symmetry.** Consider the matrices

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 \\ -1 & 0 \\ 1 & 4 \end{bmatrix}.$$

- (a) Give the sizes of AB , $(AB)^T$, $B^T A^T$, and $A^T B^T$. A student claims that $(AB)^T = A^T B^T$. Without computing any entries, explain why this cannot be true for these matrices, even though $A^T B^T$ is defined.
- (b) Compute AB and $B^T A^T$, and check that $(AB)^T = B^T A^T$.
- (c) Compute AA^T and observe that it is symmetric. Then use the rule $(XY)^T = Y^T X^T$ to prove that MM^T is symmetric for *every* matrix M .

Solution.

- (a) A is 2×3 , B is 3×2 , A^T is 3×2 and B^T is 2×3 . Hence

- AB : $(2 \times \underline{3})(\underline{3} \times 2) \rightarrow 2 \times 2$, so $(AB)^T$ is also 2×2 ;
- $B^T A^T$: $(2 \times \underline{3})(\underline{3} \times 2) \rightarrow 2 \times 2$;
- $A^T B^T$: $(3 \times \underline{2})(\underline{2} \times 3) \rightarrow 3 \times 3$.

$(AB)^T$ is 2×2 but $A^T B^T$ is 3×3 . Matrices of different sizes can never be equal, so the claim is false. Only the reversed order $B^T A^T$ has the right size.

- (b) Computing AB :

$$AB = \begin{bmatrix} 1(2) + 2(-1) + (-1)(1) & 1(1) + 2(0) + (-1)(4) \\ 0(2) + 1(-1) + 3(1) & 0(1) + 1(0) + 3(4) \end{bmatrix} = \begin{bmatrix} -1 & -3 \\ 2 & 12 \end{bmatrix}, \quad (AB)^T = \begin{bmatrix} -1 & 2 \\ -3 & 12 \end{bmatrix}.$$

With $B^T = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 0 & 4 \end{bmatrix}$ and $A^T = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ -1 & 3 \end{bmatrix}$:

$$B^T A^T = \begin{bmatrix} 2(1) + (-1)(2) + 1(-1) & 2(0) + (-1)(1) + 1(3) \\ 1(1) + 0(2) + 4(-1) & 1(0) + 0(1) + 4(3) \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ -3 & 12 \end{bmatrix} = (AB)^T \checkmark.$$

- (c) Computing AA^T :

$$AA^T = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 1+4+1 & 0+2-3 \\ 0+2-3 & 0+1+9 \end{bmatrix} = \begin{bmatrix} 6 & -1 \\ -1 & 10 \end{bmatrix},$$

which is symmetric. (Entry (i, j) is the dot product of row i and row j of A , and the dot product does not care about the order of the two rows.)

Proof. For any $m \times n$ matrix M , the product MM^T is $m \times m$ (square), and applying $(XY)^T = Y^T X^T$ with $X = M$, $Y = M^T$:

$$(MM^T)^T = (M^T)^T M^T = MM^T,$$

since transposing twice gives back M . So MM^T equals its own transpose, i.e. it is symmetric. (The same argument shows $M^T M$ is symmetric.)

3. Powers at Work: Movements Between Employment and Unemployment. A labour-market panel study follows the same group of young adults month after month. Each month, 90% of those who are employed stay employed and 10% become unemployed, while 40% of those who are unemployed find a job and 60% remain unemployed. With the states ordered as (Employed, Unemployed), this gives the **transition matrix**

$$P = \begin{bmatrix} 0.90 & 0.40 \\ 0.10 & 0.60 \end{bmatrix} \quad (\text{column } j = \text{state this month, row } i = \text{state next month}).$$

If $\mathbf{x}_k$ holds the shares of the group that are employed and unemployed in month k , then $\mathbf{x}_{k+1} = P\mathbf{x}_k$.

- (a) Interpret the entry p_{21} . Explain why each column of P must add up to 1.
- (b) Compute P^2 . Interpret the entry $(P^2)_{12}$, and explain why it is *not* equal to $(p_{12})^2 = 0.16$.
- (c) In month 0, 60% of the group are employed: $\mathbf{x}_0 = [0.60 \ 0.40]^T$. Compute $\mathbf{x}_2$ in two ways, as $P(P\mathbf{x}_0)$ and as $P^2\mathbf{x}_0$. Which property of matrix multiplication guarantees the same answer? If you had to forecast one year ahead for many different starting groups, which way would you choose?
- (d) Show that $\mathbf{q} = [0.8 \ 0.2]^T$ satisfies $P\mathbf{q} = \mathbf{q}$, and deduce that $P^k\mathbf{q} = \mathbf{q}$ for every $k \geq 1$. Interpret this result, and compare it with the shares you found in part (c).

Solution.

- (a) $p_{21} = 0.10$ is in column 1 (employed this month) and row 2 (unemployed next month): 10% of those who are employed this month will be unemployed next month.
Column j lists where the people currently in state j go next month. Each person ends up in exactly one of the two states, so the shares in a column must add up to $100\% = 1$.
- (b) Computing P^2 :

$$P^2 = \begin{bmatrix} 0.90 & 0.40 \\ 0.10 & 0.60 \end{bmatrix} \begin{bmatrix} 0.90 & 0.40 \\ 0.10 & 0.60 \end{bmatrix} = \begin{bmatrix} 0.81 + 0.04 & 0.36 + 0.24 \\ 0.09 + 0.06 & 0.04 + 0.36 \end{bmatrix} = \begin{bmatrix} 0.85 & 0.60 \\ 0.15 & 0.40 \end{bmatrix}.$$

$(P^2)_{12} = 0.60$: of those unemployed now, 60% are employed **two months** later.

It is not $(p_{12})^2 = 0.40^2 = 0.16$, because the entry of a matrix power is a row-by-column dot product, not the square of one entry. The dot product adds up all routes from “unemployed” to “employed” in two steps:

$$(P^2)_{12} = \underbrace{p_{11}p_{12}}_{U \rightarrow E \rightarrow E} + \underbrace{p_{12}p_{22}}_{U \rightarrow U \rightarrow E} = 0.90(0.40) + 0.40(0.60) = 0.36 + 0.24 = 0.60.$$

The number $(p_{12})^2$ would mean “finding a job from unemployment twice in a row”, which makes no sense: after the first month that person is already employed. Powers can be taken entry by entry only for *diagonal* matrices.

- (c) **Step by step:**

$$\begin{aligned} \mathbf{x}_1 &= P\mathbf{x}_0 = \begin{bmatrix} 0.90(0.60) + 0.40(0.40) \\ 0.10(0.60) + 0.60(0.40) \end{bmatrix} = \begin{bmatrix} 0.70 \\ 0.30 \end{bmatrix}, \\ \mathbf{x}_2 &= P\mathbf{x}_1 = \begin{bmatrix} 0.90(0.70) + 0.40(0.30) \\ 0.10(0.70) + 0.60(0.30) \end{bmatrix} = \begin{bmatrix} 0.75 \\ 0.25 \end{bmatrix}. \end{aligned}$$

Using the power:

$$P^2\mathbf{x}_0 = \begin{bmatrix} 0.85(0.60) + 0.60(0.40) \\ 0.15(0.60) + 0.40(0.40) \end{bmatrix} = \begin{bmatrix} 0.51 + 0.24 \\ 0.09 + 0.16 \end{bmatrix} = \begin{bmatrix} 0.75 \\ 0.25 \end{bmatrix} \checkmark.$$

The two results agree because of **associativity**: $P(P\mathbf{x}_0) = (PP)\mathbf{x}_0 = P^2\mathbf{x}_0$. In general $\mathbf{x}_k = P^k\mathbf{x}_0$. For a one-year forecast for many starting groups, compute P^{12} **once**; each forecast is then a single product $P^{12}\mathbf{x}_0$ instead of 12 successive products per group.

- (d) Computing:

$$P\mathbf{q} = \begin{bmatrix} 0.90(0.8) + 0.40(0.2) \\ 0.10(0.8) + 0.60(0.2) \end{bmatrix} = \begin{bmatrix} 0.72 + 0.08 \\ 0.08 + 0.12 \end{bmatrix} = \begin{bmatrix} 0.8 \\ 0.2 \end{bmatrix} = \mathbf{q}.$$

Using associativity repeatedly, $P^k\mathbf{q} = P^{k-1}(P\mathbf{q}) = P^{k-1}\mathbf{q} = \dots = P\mathbf{q} = \mathbf{q}$.

Interpretation. If 80% of the group is employed, the shares stay at 80%/20% forever: $\mathbf{q}$ is an

equilibrium (steady state). The flows balance: 10% of the 80% employed lose their job (8% of the group), while 40% of the 20% unemployed find one (also 8%). In part (c) the employed share moved $0.60 \rightarrow 0.70 \rightarrow 0.75$, i.e. it is approaching the equilibrium value 0.80.