

Linear Algebra — Class Exercise 9

Inverses and Matrix Equations

1. **One Invertible, One Singular.** Consider the matrices

$$A = \begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} -6 & 9 \\ 4 & -6 \end{bmatrix}.$$

- (a) Compute $\det(A)$ and $\det(B)$, and decide which of the two matrices is invertible.
- (b) Find the inverse of the invertible matrix. Verify your answer by computing the product with the original matrix in *both* orders.
- (c) For the singular matrix, find a non-zero vector $\mathbf{x}$ with $B\mathbf{x} = \mathbf{0}$. Use this vector to explain why B cannot have an inverse. (*Hint:* what would left-multiplying $B\mathbf{x} = \mathbf{0}$ by B^{-1} give?)

2. **Invertibility with a Parameter.** For each real number k , let

$$M = \begin{bmatrix} k-2 & 3 \\ 2 & k-1 \end{bmatrix}.$$

- (a) Find all values of k for which M is invertible. State your conclusion in a sentence that covers every value of k .
- (b) Find M^{-1} when $k = 0$, and verify it.
- (c) Take the *largest* value of k for which M is singular. Write down M for that k , and show that the system $M\mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ has no solution. Why is this consistent with what you found in part (a)?

3. **Inverses of Products and of Sums.** Consider the matrices

$$A = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}.$$

- (a) Find A^{-1} and B^{-1} .
- (b) Compute AB , and then find $(AB)^{-1}$ directly with the 2×2 formula.
- (c) Compute $B^{-1}A^{-1}$ and $A^{-1}B^{-1}$. Which one equals $(AB)^{-1}$? Explain, with a short calculation using associativity, why the order has to be reversed.
- (d) Compute $A + B$ and $(A + B)^{-1}$. Compare with $A^{-1} + B^{-1}$. What do you conclude?

4. **Solving for a Matrix.** Let

$$D = \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix}, \quad E = \begin{bmatrix} 2 & 0 \\ 2 & 4 \end{bmatrix}.$$

Solve the equation $DX = X + E$ for the unknown 2×2 matrix X . Say at every step whether you left-multiply or right-multiply, and check your answer.