

Linear Algebra — Class Exercise 9

Inverses and Matrix Equations

1. **One Invertible, One Singular.** Consider the matrices

$$A = \begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} -6 & 9 \\ 4 & -6 \end{bmatrix}.$$

- (a) Compute $\det(A)$ and $\det(B)$, and decide which of the two matrices is invertible.
- (b) Find the inverse of the invertible matrix. Verify your answer by computing the product with the original matrix in *both* orders.
- (c) For the singular matrix, find a non-zero vector $\mathbf{x}$ with $B\mathbf{x} = \mathbf{0}$. Use this vector to explain why B cannot have an inverse. (*Hint:* what would left-multiplying $B\mathbf{x} = \mathbf{0}$ by B^{-1} give?)

Solution.

- (a) $\det(A) = 7(2) - 3(4) = 14 - 12 = 2 \neq 0$, so A is **invertible**.
 $\det(B) = (-6)(-6) - 9(4) = 36 - 36 = 0$, so B is **singular**.
- (b) Swap the diagonal entries, negate the off-diagonal entries, divide by $\det(A) = 2$:

$$A^{-1} = \frac{1}{2} \begin{bmatrix} 2 & -3 \\ -4 & 7 \end{bmatrix} = \begin{bmatrix} 1 & -\frac{3}{2} \\ -2 & \frac{7}{2} \end{bmatrix}.$$

Check in both orders:

$$AA^{-1} = \begin{bmatrix} 7(1) + 3(-2) & 7(-\frac{3}{2}) + 3(\frac{7}{2}) \\ 4(1) + 2(-2) & 4(-\frac{3}{2}) + 2(\frac{7}{2}) \end{bmatrix} = \begin{bmatrix} 7-6 & -\frac{21}{2} + \frac{21}{2} \\ 4-4 & -6+7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$A^{-1}A = \begin{bmatrix} 1(7) - \frac{3}{2}(4) & 1(3) - \frac{3}{2}(2) \\ -2(7) + \frac{7}{2}(4) & -2(3) + \frac{7}{2}(2) \end{bmatrix} = \begin{bmatrix} 7-6 & 3-3 \\ -14+14 & -6+7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

- (c) The second column of B is $-\frac{3}{2}$ times the first, so $\frac{3}{2}(\text{col}_1) + 1(\text{col}_2) = \mathbf{0}$. Doubling to clear the fraction, $\mathbf{x} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ works:

$$B\mathbf{x} = \begin{bmatrix} -6(3) + 9(2) \\ 4(3) - 6(2) \end{bmatrix} = \begin{bmatrix} -18 + 18 \\ 12 - 12 \end{bmatrix} = \mathbf{0}.$$

Suppose B had an inverse. Left-multiplying $B\mathbf{x} = \mathbf{0}$ by B^{-1} would give $B^{-1}B\mathbf{x} = B^{-1}\mathbf{0}$, that is $\mathbf{x} = \mathbf{0}$. But $\mathbf{x} \neq \mathbf{0}$, a contradiction. So B has no inverse.

In words: B sends both $\mathbf{x}$ and $\mathbf{0}$ to the same vector $\mathbf{0}$, and no matrix can send $\mathbf{0}$ back to two different vectors at once.

2. **Invertibility with a Parameter.** For each real number k , let

$$M = \begin{bmatrix} k-2 & 3 \\ 2 & k-1 \end{bmatrix}.$$

- (a) Find all values of k for which M is invertible. State your conclusion in a sentence that covers every value of k .

- (b) Find M^{-1} when $k = 0$, and verify it.
- (c) Take the *largest* value of k for which M is singular. Write down M for that k , and show that the system $M\mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ has no solution. Why is this consistent with what you found in part (a)?

Solution.

(a)

$$\det(M) = (k-2)(k-1) - 3(2) = k^2 - 3k + 2 - 6 = k^2 - 3k - 4 = (k-4)(k+1).$$

M is invertible exactly when $\det(M) \neq 0$. M is invertible for every real k except $k = 4$ and $k = -1$; at those two values it is singular.

- (b) At $k = 0$: $M = \begin{bmatrix} -2 & 3 \\ 2 & -1 \end{bmatrix}$ and $\det(M) = (0-4)(0+1) = -4$ (directly: $2 - 6 = -4$). Then

$$M^{-1} = -\frac{1}{4} \begin{bmatrix} -1 & -3 \\ -2 & -2 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{3}{4} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

$$\text{Check: } MM^{-1} = \begin{bmatrix} -2(\frac{1}{4}) + 3(\frac{1}{2}) & -2(\frac{3}{4}) + 3(\frac{1}{2}) \\ 2(\frac{1}{4}) - 1(\frac{1}{2}) & 2(\frac{3}{4}) - 1(\frac{1}{2}) \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} + \frac{3}{2} & -\frac{3}{2} + \frac{3}{2} \\ \frac{1}{2} - \frac{1}{2} & \frac{3}{2} - \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

- (c) The largest singular value is $k = 4$, where $M = \begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix}$: both rows are equal. Row-reduce the augmented matrix:

$$\left[\begin{array}{cc|c} 2 & 3 & 1 \\ 2 & 3 & 2 \end{array} \right] \xrightarrow{R_2 \rightarrow R_2 - R_1} \left[\begin{array}{cc|c} 2 & 3 & 1 \\ 0 & 0 & 1 \end{array} \right].$$

The last row reads $0 = 1$, so the system is **inconsistent**. (The two equations say that the same quantity $2x + 3y$ equals both 1 and 2.)

This is consistent with part (a): if M were invertible, $\mathbf{x} = M^{-1} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ would be a solution, for *every* right-hand side. A right-hand side with no solution can only occur when M is singular, and $k = 4$ is one of the two singular values.

3. Inverses of Products and of Sums. Consider the matrices

$$A = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}.$$

- (a) Find A^{-1} and B^{-1} .
- (b) Compute AB , and then find $(AB)^{-1}$ directly with the 2×2 formula.
- (c) Compute $B^{-1}A^{-1}$ and $A^{-1}B^{-1}$. Which one equals $(AB)^{-1}$? Explain, with a short calculation using associativity, why the order has to be reversed.
- (d) Compute $A + B$ and $(A + B)^{-1}$. Compare with $A^{-1} + B^{-1}$. What do you conclude?

Solution.

- (a) $\det(A) = 3 - 2 = 1$ and $\det(B) = 6 - 5 = 1$, so dividing by the determinant changes nothing:

$$A^{-1} = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}, \quad B^{-1} = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}.$$

(b)

$$AB = \begin{bmatrix} 3(2) + 1(5) & 3(1) + 1(3) \\ 2(2) + 1(5) & 2(1) + 1(3) \end{bmatrix} = \begin{bmatrix} 11 & 6 \\ 9 & 5 \end{bmatrix}.$$

$$\det(AB) = 55 - 54 = 1, \text{ so } (AB)^{-1} = \begin{bmatrix} 5 & -6 \\ -9 & 11 \end{bmatrix}.$$

(c)

$$B^{-1}A^{-1} = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 3+2 & -3-3 \\ -5-4 & 5+6 \end{bmatrix} = \begin{bmatrix} 5 & -6 \\ -9 & 11 \end{bmatrix} = (AB)^{-1} \checkmark,$$

$$A^{-1}B^{-1} = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix} = \begin{bmatrix} 3+5 & -1-2 \\ -6-15 & 2+6 \end{bmatrix} = \begin{bmatrix} 8 & -3 \\ -21 & 8 \end{bmatrix} \neq (AB)^{-1}.$$

The reversed product is the inverse because, by associativity, the inner factors meet their own inverses first:

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I.$$

In the other order, $(AB)(A^{-1}B^{-1}) = ABA^{-1}B^{-1}$: B stands next to A^{-1} , not next to B^{-1} , so nothing cancels. (Undoing “first B , then A ” means undoing A first.)

(d)

$$A + B = \begin{bmatrix} 5 & 2 \\ 7 & 4 \end{bmatrix}, \quad \det(A + B) = 20 - 14 = 6, \quad (A + B)^{-1} = \frac{1}{6} \begin{bmatrix} 4 & -2 \\ -7 & 5 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{7}{6} & \frac{5}{6} \end{bmatrix}.$$

But

$$A^{-1} + B^{-1} = \begin{bmatrix} 1+3 & -1-1 \\ -2-5 & 3+2 \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ -7 & 5 \end{bmatrix} \neq (A + B)^{-1}.$$

$$\text{Check: } (A + B)(A^{-1} + B^{-1}) = \begin{bmatrix} 5 & 2 \\ 7 & 4 \end{bmatrix} \begin{bmatrix} 4 & -2 \\ -7 & 5 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix} = 6I \neq I.$$

Conclusion: $(A + B)^{-1} \neq A^{-1} + B^{-1}$ in general. The inverse behaves well with products (reversing the order), but there is no rule for the inverse of a sum.

4. Solving for a Matrix. Let

$$D = \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix}, \quad E = \begin{bmatrix} 2 & 0 \\ 2 & 4 \end{bmatrix}.$$

Solve the equation $DX = X + E$ for the unknown 2×2 matrix X . Say at every step whether you left-multiply or right-multiply, and check your answer.

Solution. Collect the X terms on the left: $DX - X = E$. In both terms X stands on the right, so factor it out on the right, writing $X = IX$:

$$(D - I)X = E, \quad D - I = \begin{bmatrix} 2 & 2 \\ 2 & 1 \end{bmatrix}, \quad \det(D - I) = 2 - 4 = -2 \neq 0.$$

(Writing $(D - 1)X$ would be meaningless: 1 is a number and D is a matrix.) Since $D - I$ is invertible, $(D - I)^{-1} = -\frac{1}{2} \begin{bmatrix} 1 & -2 \\ -2 & 2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & 1 \\ 1 & -1 \end{bmatrix}$. The factor $D - I$ stands to the *left* of X , so **left**-multiply both sides by its inverse:

$$X = (D - I)^{-1}E = \begin{bmatrix} -\frac{1}{2} & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} -1+2 & 0+4 \\ 2-2 & 0-4 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 0 & -4 \end{bmatrix}.$$

The order matters: right-multiplying instead would give $E(D - I)^{-1} = \begin{bmatrix} -1 & 2 \\ 3 & -2 \end{bmatrix}$, which is not a solution.

$$\text{Check: } DX = \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 0 & -4 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 2 & 0 \end{bmatrix} \text{ and } X + E = \begin{bmatrix} 1+2 & 4+0 \\ 0+2 & -4+4 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 2 & 0 \end{bmatrix} \checkmark.$$