Assignment 3 - Solutions

1. **Koala**

   In 2006, thieves planning to steal a koala from a zoo in Australia, had to change their minds after it proved too vicious to be kidnapped. After getting thoroughly mauled, they gave up and stole a crocodile instead.

   (a) A witness reported that a car seen speeding away from the zoo had a number plate that began with a V or W, followed by the digits $4, 7$, and 8 in some order, and ending with letters A, C, and E in some order. Determine the number of cars that would need to be examined to ensure that the suspect's vehicle is included.

   Solution

   Let $E =$ the number of license plates that need to be checked.

   $$n(E) = 2 \times 3! \times 3! = 72$$

   (b) In other parts of Australia, a licence plate consists of a sequence of seven symbols: number, letter, letter, letter, number, number, number, where a letter is any one of 26 letters (A - Z) and a number is one of $(0-9)$. Assume that all licence plates are equally likely.

   i. What is the probability that all symbols are different?

   Solution

   Let $S =$ be the sample space of all license plates that contain seven symbols.

   $$n(S) = 10^4 \times 26^3 = 17\,576\,000$$

   $E =$ the number of license plates whose symbols are all different.

   $$n(E) = 10 \times 26 \times 25 \times 24 \times 9 \times 8 \times 7 = P_4^{10} \cdot P_3^{26} = 78\,624\,000$$

   Therefore,

   $$P(E) = \frac{n(E)}{n(S)} = \frac{78\,624\,000}{175\,760\,000} = \frac{378}{845} = 0.4473$$

   ii. What is the probability that all symbols are different and the first number is the largest among the numbers?

   Solution

Let L = the largest number in the license plate and S denote the other numbers (which
are smaller than L.

There are four positions in which the largest number can appear:

$$L\_\_\_S\,S\,S \qquad S\_\_\_L\,S\,S \qquad S\_\_\_S\,L\,S \qquad S\_\_\_S\,S\,L$$

Since the largest number appears in position #1 in only one of the four arrangements, this means that the number of plates that have all symbols which are different in which the first number is the largest among the numbers is $78\,642\,000 \div 4$ or $19\,656\,000$

$$\Rightarrow P(L) = \frac{P(E)}{4} = \frac{19\,656\,000}{175\,760\,000} = \frac{189}{1\,690} = 0.1118$$

2. **Cards**

Shuffle a deck of 52 cards. What is the probability that

(a) the top card is a heart?

Solution

Let S = the number of cards in a deck
H = the event that the top card is a heart.

$$P(H) = \frac{n(H)}{n(S)} = \frac{C_1^{13}}{52} = \frac{13}{52} = \frac{1}{4} = 0.25$$

(b) all cards of the same suit end up next to each other?

Solution

Let S = the number of ways to arrange 52 cards in a row
G = the event that the same suits end up next to each other.

$n(S) = 52!$; number of ways to arrange all of the cards in a row: 52!

$n(G) = (13!)^4 \cdot 4!$; number of ways to arrange a single suit: 13!
; number of ways that we can arrange the blocks of suits: 4!

Therefore,

$$P(G) = \frac{n(G)}{n(S)} = \frac{(13!)^4 \cdot 4!}{52!} = 4.4739 \times 10^{-28}$$

(c) the diamonds are together?

Solution

Let $D =$ the event that the diamonds are together.

$n(D) = 13! \cdot 39! \cdot 40$; number of ways of arranging the diamonds to appear together: 13!
; number of organizing ways to organize the other cards: 39!
; number of positions that the block of diamonds can occupy: 40

Therefore,

$$P(D) = \frac{n(D)}{n(S)} = \frac{13! \cdot 39! \cdot 40}{52!} = 6.2991 \times 10^{-11}$$

3. **Committee**

Four people are chosen from a group of ten persons consisting or four men and six women to serve on a committee. Three of the women are sisters. What is the probability that the four people chosen will

(a) consists of four women?

Solution

Let $W =$ the number of women on the committee of four people
$n(S) = C_4^{10}$; number of ways of choosing 4 people out of 10
$n(W) = C_4^6 \cdot C_0^4$; there are 6 women, and we need 4 of them
; there are 4 men, we need none of them

Therefore,

$$P(W = 4) = \frac{C_4^6 \cdot C_0^4}{C_4^{10}} = \frac{15}{210} = \frac{1}{14} = 0.0714$$

(b) consists of two men and two women?

Solution

$n(W) = C_2^6 \cdot C_2^4$; there are 6 women, and we need 4 of them
; there are 4 men, we need 2 of them

Therefore,

$$P(W = 4) = \frac{n(W)}{n(S)} = \frac{C_2^6 \cdot C_2^4}{C_4^{10}} = \frac{90}{210} = \frac{3}{7} = 0.4286$$

(c) consists of more women than men?

Solution

$$
\begin{aligned}
P(W > M) = P(W = 3) + P(W = 4) &= \frac{C_3^6 \cdot C_1^4}{C_4^{10}} + \frac{C_4^6 \cdot C_0^4}{C_4^{10}} \\
&= \frac{80}{210} + \frac{15}{210} \\
&= \frac{19}{42} = 0.4524
\end{aligned}
$$

(d) include the three sisters?

Solution

Let K = the committee consists of the three sisters

$n(K) = C_3^3 \cdot C_1^7$; there are 3 women who are sisters, and we need all of them
; there are 7 people (not related to the sisters), we need 1 of them

Therefore,

$$P(K) = \frac{n(K)}{n(S)} = \frac{C_3^3 \cdot C_1^7}{C_4^{10}} = \frac{7}{210} = \frac{1}{30} = 0.0333$$

4. **Tomatoes v. Caterpillars**

Tomato plants emit distress signals when under attack. However, some caterpillars can use the chemicals in their saliva to 'silence' them.

At a research greenhouse, there are 24 tomato plants. Four plants are selected at random for testing without replacement. Suppose that six of the plants have been attacked by 'silencing' caterpillars whose saliva suppresses distress signals.

(a) What is the probability that exactly one of the selected plants has been attacked by a silent caterpillar?

Solution

Let S = the number of ways that four plant can be chosen out of 24.
A = the number of plants in our sample that have been attacked by a caterpillar .

$n(S) = C_4^{24}$; number of ways the choose four plants out of 24: C_4^{24}
$n(C) = C_1^6 \cdot C_3^{18}$; there are six plants that have been attacked and we just need one: C_1^6
; the other 18 plants are unaffected, and we need three: C_3^{18}

Therefore,

$$P(A) = \frac{n(A)}{n(S)} = \frac{C_1^6 \cdot C_3^{18}}{C_4^{24}} = \frac{4\,896}{10\,626} = \frac{816}{1\,771} = 0.4607$$

(b) What is the probability that at least one of the selected plants has been attacked by a silent caterpillar?

Solution

At least one plant has been attacked. This means we either have one plant, two plants, three plants, or four plants in our sample has been attacked.

Then,

$$P(A \geq 1) = \frac{C_1^6 \cdot C_3^{18}}{C_4^{24}} + \frac{C_2^6 \cdot C_2^{18}}{C_4^{24}} + \frac{C_3^6 \cdot C_1^{18}}{C_4^{24}} + \frac{C_4^6 \cdot C_0^{18}}{C_4^{24}} = \frac{7\,566}{10\,626} = \frac{1\,261}{1\,771} = 0.7120$$

Alternatively, we can do

$$P(A \geq 1) = 1 - P(A = 0) = 1 - \frac{C_0^6 \cdot C_4^{18}}{C_4^{24}} = 1 - \frac{3\,060}{10\,626} = \frac{1\,216}{1\,771} = 0.7120$$

(c) In addition to the six plants attacked by silent caterpillars, four different plants have been infected by a fungus that prevents them from releasing distress signals altogether. What is the probability that exactly one selected plant has been attacked by a silent caterpillar, and exactly one selected plant has been infected by the fungus?

Solution

Let $FA =$ one plant has been infected by a fungus **and** one plant has been silenced by a caterpillar

Then

$$P(FA) = \frac{n(FA)}{n(S)} = \frac{C_1^4 \cdot C_1^6 \cdot C_2^{14}}{C_4^{24}} = \frac{2\,184}{10\,626} = \frac{52}{253} = 0.2055$$

5. **Passwords**

A security system requires users to create a password that is eight characters long, and each character is one of: 26 lower case letters (a-z), 26 upper case letters (A-Z), and 10 digits (0-9).

(a) What is the probability that a randomly selected password consists of only lower case letters (repetition allowed).

Solution

Total number of passwords:

$$n(S) = 62^8$$

Let LC = password consists of only lower case letters
$n(LC) = 26^8$

$$P(LC) = \frac{n(LC)}{n(S)} = \frac{26^8}{62^8} = \frac{13^8}{31^8} = 9.5643 \times 10^{-4}$$

(b) What is the probability that a randomly selected password starts with 4 letters, followed by 4 digits (repetition not allowed).

Solution

Let G = password consists of four letters, followed by four digits
$n(G) = 52 \cdot 51 \cdot 50 \cdot 49 \cdot 10 \cdot 9 \cdot 8 \cdot 7 = P_4^{52} \cdot P_4^{10}$

$$P(G) = \frac{n(G)}{n(S)} = \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 10 \cdot 9 \cdot 8 \cdot 7}{62^8} = \frac{P_4^{52} \cdot P_4^{10}}{62^8} = 1.4998 \times 10^{-4}$$

(c) What is the probability that a randomly generated password contains only letters with the sequence 'HINT' in consecutive order, regardless of letter case? (repetition allowed

Solution

Case 1: Let X = the password is $HINThint$

Total number of passwords that look like $HINThint$

$$n(X) = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 1 \cdot 1 \cdot 1 \cdot 1 = 2^4$$

Case 2: Let K = the password is $HINT****$

$n(K) = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 52^4 \cdot 5$; number of ways to spell HINT in both cases: 2^4
; remaining 4 characters has to be letters; repetition allowed: 52^4
; number of spots that HINT can be positioned in the password: 5

Let H = the password contains the sequence $HINT$ in it

$$P(H) = \frac{n(K) - n(X)}{n(S)} = \frac{2^4 \cdot 52^4 \cdot 5 - 2^4}{62^8} = \frac{584929264}{62^8} = 2.679 \times 10^{-6}$$

(d) What is the probability that a randomly generated password contains a mix of four digits and the sequence 'HINT' in consecutive order, regardless of letter case? (repetition not allowed).

Solution

Let H = the password contains the sequence HINT and a mix of four digits; repetition not allowed.

$$n(H) = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 5$$

; number of ways to spell HINT (upper/lower): 2^4
; remaining 4 characters has to be digits and
; repetition not allowed: P_4^{10}
; number of spots that HINT can
; be positioned in the password: 5

$$P(H) = \frac{n(H)}{n(S)} = \frac{2^4 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 5}{62^8} = \frac{2^4 \cdot P_4^{10} \cdot 5}{62^8} = \frac{403200}{62^8} = \frac{584929296}{62^8} = 1.8467 \times 10^{-9}$$

(e) How many passwords contain exactly one 'X' (in either upper or lower case) among the first three characters, assuming no repetition of characters is allowed?

Solution

We need to place exactly one 'X' or 'x' in one of the first three slots. We have 3 choices for the slot and 2 choices for which of 'X', 'x' is in this slot. Next we have to select two more characters for the remaining two slots in the first 3 slots, that is P_2^{60} choices. Finally we have to select the characters for the 5 slots at the back: we have P_5^{59} choices.

Therefore, the total number of passwords with exactly one X or x in the first 3 slots is:

$$n(X) + n(x) = C_1^3 \cdot C_1^2 \cdot P_2^{60} \cdot P_5^{59} = 1.2752 \times 10^{13}$$