Assignment 5 - Solutions

1. **Bread Guilds**

   In the 14th century, bakers of white and brown bread split into two rival guilds and didn't reunite until the 17th century.

   A recent study found that 45% of people prefer white bread, while the rest prefer brown bread, Suppose that 10 people are randomly selected.

   (a) What is the probability that two or three of them prefer white bread?

   Solution

   Let X = the number of people in the group who prefers white bread

   $X \sim$ Binomial with $n = 10$ and $p = 0.45$

$$\begin{aligned} P(X = 2 \cup X = 3) &= P(X = 2) + P(X = 3) \\ &= C_2^{10}(0.45)^2(1 - 0.45)^8 + C_3^{10}(0.45)^3(1 - 0.45)^7 \\ &= 0.0763 + 0.1665 \\ &= 0.2428 \end{aligned}$$

   (b) What is the probability that at least one person prefers white bread?

   Solution

$$\begin{aligned} P(X \geq 1) &= P(X = 1) + P(X = 2) + P(X = 3) + \cdots + P(X = 10) \\ &= 1 - P(X = 0) \\ &= 1 - C_0^{10}(0.45)^0(1 - 0.45)^{10} \\ &= 1 - 0.0025 \\ &= 0.9975 \end{aligned}$$

   (c) What is the probability that fewer than four people prefer brown bread?

   Solution

   Let Y = the number of people in the group who prefers brown bread

   $Y \sim$ Binomial with $n = 10$ and $p = 0.55$

$$\begin{aligned} P(Y < 4) &= P(Y = 0) + P(Y = 1) + P(Y = 2) + P(Y = 3) \\ &= C_0^{10}(0.55)^0(0.45)^{10} + C_1^{10}(0.55)(0.45) + C_2^{10}(0.55)^2(0.45)^8 + C_3^{10}(0.55)^3(0.45)^7 \\ &= 0.0003 + 0.0042 + 0.0229 + 0.0746 \\ &= 0.1020 \end{aligned}$$

(d) What is the expected number of people in the group that prefer brown bread? What is the standard deviation?

Solution

$$\mathbb{E}[Y] = n \cdot p = 10(0.55) = 5.5$$

$$\mathbb{S}[Y] = \sqrt{n \cdot p \cdot (1-p)} = \sqrt{10(0.55)(1-0.55)} = \sqrt{2.475} = 1.5732$$

2. **Gut Biome**

Not to be tried at home, but doctors in Finland believe that feeding babies small amounts of their mother's poo may strengthen their gut bacteria.

In a pilot study, researchers measured the time (in hours) it took for infants' gut microbiomes to reach a "balanced" state after the treatment. Suppose this time follows a normal distribution with a mean of 72 hours and a standard deviation of 10 hours.

(a) What is the probability that a randomly selected baby achieves a balanced gut microbiome in less than 60 hours?

Solution

Let X = the time it takes for baby to achieve a balanced gut microbiome
$X \sim$ Normal with $\mu = 72$ and $\sigma = 10$

$$P(X < 60) = P\left(Z < \frac{60-72}{10}\right) = P(Z < -1.20) = 0.1151$$

(b) What is the probability that the balance time is more than 85 hours?

Solution

$$P(X > 85) = P\left(Z > \frac{85-72}{10}\right) = P(Z > 1.30) = 1 - P(Z < 1.30) = 1 - 0.9032 = 0.0968$$

(c) What is the probability that the balance time is between 70 and 80 hours?

Solution

$$\begin{aligned} P(70 < X < 80) = P\left(\frac{70-72}{10} < Z < \frac{80-72}{10}\right) &= P(-0.20 < Z < 0.80) \\ &= P(Z < 0.80) - P(Z < -0.20) \\ &= 0.7881 - 0.4207 \\ &= 0.3674 \end{aligned}$$

(d) Only the slowest 10% of babies take this long to adjust. What is the minimum number of hours required to be in this slowest 10% group?

Solution

The "slowest 10%" means the upper 10% of the distribution. Want k such that $P(Z > k) = 0.10$ (ie. the area to the right of Z is 0.10) $\Rightarrow$ $Z = +1.28$

$$\begin{aligned} X &= Z\sigma + \mu \\ &= +1.28(10) + 72 \\ &= 84.8\, hours \end{aligned}$$

3. **Thisbe and Pyramus**

In Greek mythology, Pyramus and Thisbe were lovers separated by a wall. Today, the Pyramus and Thisbe Society is a group of lawyers who specialise in wall-related disputes.

A recent report revealed that the probability a randomly selected lawyer in Greece belongs to this society is 0.40.

(a) In a small firm of 6 lawyers, what is the probability that 4 or 5 of them belong to the Pyramus and Thisbe Society?

Solution

Let X = the number of lawyers in the group that belong to the society

$X \sim$ Binomial with $n = 6$ and $p = 0.40$

$n \cdot p = 6 \cdot (0.40) = 2.4 < 5 \quad \Rightarrow \quad$ cannot use approximation

$$\begin{aligned} P(X = 4 \cup X = 5) &= P(X = 4) + P(X = 5) \\ &= C^6_4(0.40)^4(0.60)^2 + C^6_5(0.40)^5(0.60) \\ &= 0.1382 + 0.0369 \\ &= 0.1751 \end{aligned}$$

(b) In a larger regional sample of 80 lawyers, what is the probability that at least 20 are members of the society?

Solution

$n \cdot p = 80 \cdot (0.40) = 32 > 5 \quad ; \quad n(1-p) = 80 \cdot (0.60) = 48 > 5$

$\Rightarrow$ Normal approximation to the Binomial

$X \sim$ Normal

$\mu = np = 80(0.40) = 32$ and $\sigma = \sqrt{np(1-p)} = \sqrt{80(0.40)(0.60)} = \sqrt{19.2}$

$$\begin{aligned} P(X \geq 20) = P(X > 19.5) &= P\left(Z > \frac{19.5 - 32}{\sqrt{19.2}}\right) \\ &= P(Z > -2.85) \\ &= 1 - P(Z < -2.85) \\ &= 1 - 0.0022 \\ &= 0.9978 \end{aligned}$$

(c) In a nationwide survey of 300 lawyers, estimate the probability that between 110 and 135 (exclusive) are members.

Solution

$n \cdot p = 300 \cdot (0.40) = 120 > 5 \quad ; \quad n(1-p) = 300 \cdot (0.60) = 180 > 5$

$\Rightarrow$ Normal approximation to the Binomial

$X \sim$ Normal with
$\mu = np = 300(0.40) = 120$ and $\sigma = \sqrt{np(1-p)} = \sqrt{300(0.40)(0.60)} = \sqrt{72}$

$$\begin{aligned} P(110 < X < 135) = P(110.5 < X < 134.5) &= P\left(\frac{110.5 - 120}{\sqrt{72}} < Z < \frac{134.5 - 120}{\sqrt{72}}\right) \\ &= P(-1.12 < Z < 1.71) \\ &= P(Z < 1.71) - P(Z < -1.12) \\ &= 0.9564 - 0.1314 \\ &= 0.8250 \end{aligned}$$

(d) In a separate study of 400 lawyers, estimate the probability that fewer than 270 are not members of the society.

Solution

Let Y = the number of lawyers who do not belong in the society
$Y \sim$ Binomial with $n = 400$ and $p = 0.60$

$n \cdot p = 400 \cdot (0.60) = 240 > 5 \quad ; \quad n(1-p) = 400 \cdot (0.40) = 160 > 5$

$\Rightarrow$ Normal approximation to the Binomial

$Y \sim$ Normal with
$\mu = np = 400(0.60) = 240$ and $\sigma = \sqrt{np(1-p)} = \sqrt{400(0.60)(0.40)} = \sqrt{96}$

$$\begin{aligned} P(X < 270) = P(X < 269.5) &= P\left(Z < \frac{269.5 - 240}{\sqrt{96}}\right) \\ &= P(Z < 3.01) \\ &= 0.9987 \end{aligned}$$

4. **Goldfish**

In 2023, a Japanese YouTuber used motion tracking so his pet fish could remotely control a Nintendo Switch console and 'play' video games. Left alone for seven hours, the fish logged into the Nintendo store, set up a PayPal account and deposited 500 yen from his credit card.

Researchers later became curious about the swimming patterns of such "tech-savvy" goldfish. They found that the distance (in cm) a goldfish swims in a 5-minute period follows a normal distribution with a mean of 45 cm and a standard deviation of 6 cm.

(a) What is the probability that a randomly selected goldfish swims more than 55 cm in 5 minutes?

Solution

Let X = the distance that a goldfish swims in 5 minutes
$X \sim$ Normal with $\mu = 45$ and $\sigma = 6$

$$P(X > 55) = P\left(Z > \frac{55-45}{6}\right) = P(Z > 1.67) = 0.0475$$

(b) Suppose a goldfish has already surpassed 50 cm. What is the probability that it ends up swimming between 60 cm and 65 cm in total?

Solution

$$\begin{aligned} P(60 < X < 65 \mid X > 50) = \frac{P(60 < X < 65)}{P(X > 50)} &= \frac{P(\frac{60-45}{6} < Z < \frac{65-45}{6})}{P(Z > \frac{50-45}{6})} \\ &= \frac{P(2.5 < Z < 3.33)}{P(Z > 0.83)} \\ &= \frac{P(Z < 3.33) - P(Z < 2.5)}{P(Z > 0.83)} \\ &= \frac{0.99957 - 0.99379}{0.20233} \\ &= 0.0285 \end{aligned}$$

(c) How far must a goldfish swim to be in the top 5% of swimmers?

Solution

Want k such that $P(Z > k) = 0.05 \quad \Rightarrow \quad Z = 1.645$

$$\begin{aligned} X &= Z\sigma + \mu \\ &= 1.645(6) + 45 \\ &= 54.87\,cm \end{aligned}$$

(d) In a tank of 120 goldfish, each one attempts to "press a button" using motion tracking. Suppose the probability of success for a single fish is 0.12. What is the probability that fewer than 10 of them succeed?

Solution

Let Y = the number of goldfish that succeed in pushing the button
$Y \sim$ Binomial with $n = 120$ and $p = 0.12$

$np = 120(0.12) = 14.4 > 5 \quad n(1-p) = 120(0.88) = 105.6 > 5$
Normal approximation to the Binomial
with $\mu = np = 14.4 \quad \sigma = \sqrt{np(1-p)} = \sqrt{120(0.12(0.88)} = \sqrt{12.672}$

$$P(X < 10) = P(X < 9.5) = P\left(Z < \frac{9.5 - 14.4}{\sqrt{12.672}}\right) = P(Z < -1.38) = 0.0838$$

(e) In a smaller experiment with 10 goldfish, what is the probability that exactly 1 or 2 swim more than 55 cm in a 5-minute trial?

Solution

$$P(X > 55) = P\left(Z > \frac{55 - 45}{6}\right) = P(Z > 1.67) = 0.0478$$

Let Z = the number of goldfish that can swim more than 55 cm.
$Z \sim$ Binomial with $n = 10$ and $p = 0.0478$

$$\begin{aligned} P(Z = 1 \cup Z = 2) &= P(Z = 1) + P(Z = 2) \\ &= C_1^{10}(0.0478)(0.9522)^9 + C_2^{10}(0.0478)^2(0.9522)^8 \\ &= 0.3076 + 0.0695 \\ &= 0.3771 \end{aligned}$$

5. **Security at the Louvre**

In 1911, a journalist crept into a sarcophagus in the Louvre and spent the night there to show how bad the museum's security was.

Modern researchers studying museum security incidents found that the response time (in minutes) to detect a security breach follows a normal distribution with a mean of 42 minutes and a standard deviation of 7 minutes.

(a) What is the probability that a randomly selected incident goes undetected for more than 53 minutes?

Solution

Let X = the amount of time an incident goes undetected.
$X \sim$ Normal with $\mu = 42$ and $\sigma = 7$

$$P(X > 53) = P(Z > \left(Z > \frac{53 - 42}{7}\right) = P(Z > 1.57) = 0.0582$$

(b) If a breach has already gone undetected for at least 40 minutes, what is the probability that it lasts longer than 55 minutes?

Solution

$$\begin{aligned} P(X > 55 \mid X \geq 40) = \frac{P(X \geq 40 \cap X > 55)}{P(X > 40)} = \frac{P(X > 55)}{P(X > 40)} &= \frac{P(Z > \frac{55-42}{7})}{P(Z > \frac{40-42}{7})} \\ &= \frac{P(Z > 1.86)}{P(Z > -0.29)} \\ &= \frac{0.0314}{0.6141} \\ &= 0.0511 \end{aligned}$$

(c) Find the detection time range that captures the middle 99% of incidents.

Solution

Want k such that $P(-k < Z < k) = 0.99 \quad \Rightarrow \quad Z = \pm 2.575$

$$\begin{aligned} X &= \pm Z\sigma + \mu \\ &= \pm 2.575(7) + 42 \qquad \Rightarrow \qquad x_1 = 23.975 \quad ; \quad x_2 = 60.025 \end{aligned}$$

The detection time corresponding to the middle 99% of all incidents ranges from 23.975 minutes to 60 minutes.

(d) In a review of 120 simulated break-ins, each one is caught by a motion sensor with probability 0.93. What is the probability that fewer than 110 are detected?

Solution

Let Y = the number break ins caught by the motion sensor

$Y \sim$ Binomial with $n = 120$ and $p = 0.93$

$np = 120(0.93) = 111.6 > 5 \quad np(1-p) = 120(0.93)(0.07) = 8.4 > 5$

Normal approximation: $\mu = np = 111.6$ and $\sigma = \sqrt{120(0.93)(0.07)} = \sqrt{7.812}$

$$P(X < 110) = P(X < 109.5) = P\left(Z < \frac{109.5 - 111.6}{\sqrt{7.812}}\right) = P(Z < -0.7513) = 0.2262$$

(e) In a smaller trial of 20 museum guards, each guard independently notices suspicious activity with probability 0.8. What is the probability that exactly 14 or 16 of them spot the intruder?

Solution

$np = 20(0.8) = 16 > 5 \quad n(1-p) = 20(0.2) = 4 < 5$ cannot use approximation

$$\begin{aligned} P(Y = 14 \cup X = 16) & P(Y = 14) + P(Y = 16) \\ &= C^{20}_{14}(0.80)^{14}(0.20)^{6} + C^{20}_{16}(0.80)^{16}(0.20)^{4} \\ &= 0.1091 + 0.2182 \\ &= 0.3273 \end{aligned}$$