L10. The Normal Distribution

Example 1

Suppose that X is a random variable that is normally distributed. Compute the following probabilities:

a. $P(1.28 < Z)$

b. $P(-0.54 > Z)$

c. $P(-2.33 < Z)$

d. $P(Z > 2.33)$

e. $P(Z > 3.05)$

f. $P(-1 < Z < 1)$

g. $P(-2 < Z < 2)$

h. $P(-3 < Z < 3)$

Solution

a. $P(1.28 < Z) = 0.1003$

b. $P(0.54 < Z) = 0.7054$

c. $P(-2.33 < Z) = 0.990$

d. $P(Z > 2.33) = 0.0099$

e. $P(Z > 3.05) = 0.0011$

f. $P(-1 < Z < 1) = P(Z < 1) - P(Z < -1) = 0.8413 - 0.1587 = 0.6826$

g. $P(-2 < Z < 2) = P(Z < 2) - P(Z < -2) = 0.9772 - 0.0228 = 0.9544$

h. $P(-3 < Z < 3) = P(Z < 3) - P(Z < -3) = 0.9987 - 0.0013 = 0.9974$

Example 2

Suppose that X is a random variable that is normally distributed. Find the value of k that satisfies the statement

a. $P(k < Z) = 0.9$

b. $P(k > Z) = 0.15$

c. $P(Z < k) = 0.9451$

d. $P(Z > k) = 0.6616$

e. $P(-k < Z < k) = 0.95$

f. $P(-k < Z < k) = 0.99$

g. $P(-k < Z < k) = 0.80$

Solution

a. $P(k < Z) = 0.9 \Rightarrow k = -1.28$

b. $P(k > Z) = 0.15 \Rightarrow k = 1.04$

c. $P(Z < k) = 0.9451 \Rightarrow k = 1.6$

d. $P(Z > k) = 0.6616 \Rightarrow k = -0.41$

e. $P(-k < Z < k) = 0.95 \Rightarrow k = \pm 1.96$

f. $P(-k < Z < k) = 0.99 \Rightarrow k = \pm 2.575$

g. $P(-k < Z < k) = 0.80 \Rightarrow k = \pm 1.28$

Example 3

Suppose that X is a random variable that is normally distributed with a mean of $\mu = 10$ and a standard deviation of $\sigma = 2$. Calculate the following probabilities:

a. $P(X < 14)$

b. $P(X \leq 8)$

c. $P(X > 5)$

d. $P(7.5 < X < 11.5)$

e. $P(X < 0)$

f. $P(X \geq 22)$

Solution

Let X = a continuous random variable with $\mu = 10$ and $\sigma = 2$.

To convert X into Z values, we use

$$Z = \frac{X - \mu}{\sigma}$$

a. $P(X < 14)$

$$\begin{aligned} P(X < 14) &= P\left(Z < \frac{14-10}{2}\right) \\ &= P(Z < 2) \\ &= 0.9972 \end{aligned}$$

b. $P(X \le 8)$

$$\begin{aligned} P(X < 8) &= P\left(Z < \frac{8-10}{2}\right) \\ &= P(Z < -1.00) \\ &= 0.1587 \end{aligned}$$

c. $P(X > 5)$

$$\begin{aligned} P(X > 5) &= P\left(Z > \frac{15-10}{2}\right) \\ &= P(Z > -2.5) \\ &= 0.9938 \end{aligned}$$

d. $P(7.5 < X < 11.5)$

$$\begin{aligned} P(7.5 < X < 11.5) &= P\left(\frac{7.5-10}{2} < Z < \frac{11.5-10}{2}\right) \\ &= P(-1.25 < Z < 0.75) \\ &= P(Z < 0.75) - P(Z < -1.25) \\ &= 0.7734 - 0.1056 \\ &= 0.6678 \end{aligned}$$

e. $P(X < 0)$

$$\begin{aligned} P(X < 0) &= P\left(Z < \frac{0-10}{2}\right) \\ &= P(Z < -5) \\ &\approx 0 \end{aligned}$$

f. $P(X \ge 22)$

$$\begin{aligned} P(X \ge 22) &= P\left(Z < \frac{22-10}{2}\right) \\ &= P(Z > 6) \\ &\approx 0 \end{aligned}$$

Example 4

In 1743, King Gustav III of Sweden ordered one man to drink an excessive amount of coffee and his identical twin drink an excessive amount of tea to see which died first. Both men outlived the king.

The amount of time (in minutes) that customers spend in a coffee shop follows a normal distribution with a mean of 35 minutes and a standard deviation of 8 minutes.

a. What is the probability that a randomly selected customer spends more than 45 minutes in the coffee shop?

b. What is the probability that a customer spends between 30 and 40 minutes in the coffee shop?

c. What percentage of customers spend less than 20 minutes in the coffee shop?

d. 90% of customers spend at most how many minutes at the coffee shop?

Solution

Let X = the amount of time (in minutes) a customer spends in the coffee shop.

X is normally distributed with a mean of $\mu = 35$ and a standard deviation of $\sigma = 8$

a. Want $P(X > 45)$.

$$\begin{aligned} P(X > 45) &= P\left(Z > \frac{45-35}{8}\right) \\ &= P(Z > 1.25) \\ &= 0.1056 \end{aligned}$$

b. Want $P(30 < X < 40)$

$$\begin{aligned} P(30 < X < 140) &= P\left(\frac{30-35}{8} < Z < \frac{40-35}{8}\right) \\ &= P(-0.63 < Z < 0.63) \\ &= P(-0.63 < Z < 0.63) \\ &= 0.7357 - 0.2643 \\ &= 0.4714 \end{aligned}$$

Note: $P(-0.62 < Z < 0.62) = 0.7324 - 0.2676 = 0.4648$ would also be an acceptable answer.

c. Want $P(X < 20)$:

$$\begin{aligned} P(X < 20) &= P\left(Z < \frac{20-35}{8}\right) \\ &= P(Z < -1.88) \\ &= 0.0.0301 \end{aligned}$$

$\Rightarrow$ Approximately 3.04% of customers spend less than 20 minutes.

d. Want to find X such that $P(Z < k) = 0.90$.

The closest value of Z with 90% of the area to the left of it is $Z = 1.28$

$$Z = \frac{X-\mu}{\sigma} \quad \therefore \quad \begin{aligned} X &= Z\sigma + \mu \\ &= 1.28(8) + 35 \\ &= 45.24 \text{minutes} \end{aligned}$$

Example 5

In 2020, a startled deer in the Czech Republic snagged a hunter's rifle with its antlers and bolted into the forest, leaving the hunter both disarmed and dumbfounded. The police were called, but the deer and the rifle have yet to be recovered.

The distance (in meters) a deer bolts when startled follows a normal distribution with a mean of 300 meters and a standard deviation of 40 meters.

a. What is the probability that the deer ran more than 360 meters, making any attempt to recover the rifle even more hopeless?

b. What is the probability that it ran a distance between 280 and 320 meters?

c. Only the most elusive 5% of deer dash far enough to truly vanish without a trace. What is the minimum distance a deer must run to be in this elusive top 5%?

Solution

Let X = the distance (in meters) a deer bolts when startled.

X is a normally distributed random variable with $\mu = 300$ and $\sigma = 40$.

a. Want $P(X > 360)$

$$\begin{aligned} P(X > 360) &= P\left(Z > \frac{360-300}{40}\right) \\ &= P(Z > 1.5) \\ &= 0.0668 \end{aligned}$$

b. Want $P(280 < X < 320)$

$$
\begin{aligned}
P(208 < X < 320) &= P\left(\frac{280-300}{40} < Z < \frac{300-320}{40}\right) \\
&= P(-0.5 < Z < 0.5) \\
&= P(Z < 0.5) - P(Z < 0.5) \\
&= 0.6915 - 0.3085 \\
&= 0.3830
\end{aligned}
$$

c. Want to find X such that $P(Z > k) = 0.05$.
The closest value of Z with only 5% of the area to the right or it is $Z = 1.645$

$$
\begin{aligned}
X &= Z \cdot \sigma + \mu \\
&= 1.645 \cdot 40 + 300 \\
&= 365.8 \text{ meters}
\end{aligned}
$$

The deer must run at least 365.8 meters to be in the top 5% of the most elusive deers.

Example 6

The flag ship of Captain Morgan sank in 1671. It was discovered in 2011 by an expedition funded by the rum makers, Captain Morgan.

Let X be a normally distributed random variable representing the volume of liquor (in mL) in a standard bottle of Captain Morgan rum. According to quality control records, the mean fill volume is $750\,mL$, with a standard deviation of $6\,mL$.

a. What is the probability that a randomly selected bottle contains less than 740 mL of rum? (i.e., a bottle that might make even Captain Morgan raise an eyebrow)

b. Suppose there are 10 bottle of Captain Morgan rum on the shelves. What is the probability that fewer than three contain less than $740\,mL$ of rum in them?

c. The company promises that only the lowest 2.5% of under filled bottles should ever reach the shelves. What is the threshold volume (in mL) that separates the bottom 2.5% from the rest?

Solution

Let X = the volume of liquor (in mL) in a bottle of Captain Morgan rum.
X is a normally distributed random variable with $\mu = 750$ and $\sigma = 6$

a. Want $P(X < 740)$.

$$\begin{aligned} P(X < 740) &= P\left(Z < \frac{740 - 750}{5}\right) \\ &= P(Z < -1.67) \\ &= 0.0475 \end{aligned}$$

b. Let Y = the number of bottles on the shelf that contain less than $740\,mL$ of rum.

$\because$ the number of bottles is discrete, Y is a binomial random variable with $n = 10$, and $p = 0.0475$

Want $P(Y < 3) = P(Y = 0) + P(Y = 1) + P(Y = 2)$

$$\begin{aligned} P(Y < 3) &= P(Y = 0) + P(Y = 1) + P(Y = 2) \\ &= C_0^{10}(0.0475)^0(1 - 0.0475)^{10} + C_1^{10}(0.0475)^1(1 - 0.0475)^9 \\ &\quad + C_2^{10}(0.0475)^2(1 - 0.0475)^8 \\ &= 0.6202 + 0.3091 + 0.0624 \\ &= 0.9917 \end{aligned}$$

c. Want to find X such that $P(Z < k) = 0.025$.

The closest value of Z with 2.5% of the area underneath the curve to the left of it is $Z = -1.96$

$$\begin{aligned} X &= Z\sigma + \mu \\ &= -1.96(6) + 750 \\ &= 738.24\,mL \end{aligned}$$

So the threshold volume is approximately 738.24 mL. Only bottles below this volume fall into the bottom 2.5%.