L12. Sampling Distributions of the Mean and the Central Limit Theorem

Example 1

A bank auditor claims that credit card balances are normally distributed, with a mean of \$ 2780 and a standard deviation of \$ 900.

a. What is the probability that a randomly selected credit card holder has a credit card balance less than \$ 2500?

b. You randomly select 50 credit card holders. What is the probability that their mean credit card balance is less than \$ 2500?

c. What is the probability that for 36 randomly selected card holders, their mean credit balance is between \$ 3000 and \$3500?

Solution

a. Let X = the amount on the credit card balance.

$X \sim N(2780, 900)$

$P(X < 2500) = P(Z < -0.31) = 0.3783$

b. Want to know what is the probability that the average/mean credit card balance for these 50 people is under \$2500. So this is about a sampling distribution of the means.

Let $\overline{X}$ = the average credit card balance.

CLT: $$\overline{X} \sim N(\mu_{\overline{X}}, \sigma_{\overline{X}}) = N\left(\mu, \frac{\sigma}{\sqrt{n}}\right) = N\left(2780, \frac{900}{\sqrt{50}}\right)$$

$$P(\overline{X} < 2500) = P\left(Z < \frac{\overline{X} - \mu_{\overline{x}}}{\sigma_{\overline{X}}}\right)$$
$$= P\left(Z < \frac{2500 - 2780}{900/\sqrt{50}}\right) = P(Z < -2.20) = 0.0139$$

c. Want the probability that the mean credit card balance for a sample of 36 clients.

CLT: $$\overline{X} \sim N(\mu_{\overline{X}}, \sigma_{\overline{X}}) = N\left(\mu, \frac{\sigma}{\sqrt{n}}\right) = N\left(2780, \frac{900}{\sqrt{36}}\right)$$

$$\begin{aligned} P(3000 < \overline{X} < 3500) &= P\left(\frac{\overline{X}_1 - \mu_{\overline{x}}}{\sigma_{\overline{X}}} < Z < \frac{\overline{X}_2 - \mu_{\overline{x}}}{\sigma_{\overline{X}}}\right) \\ &= P\left(\frac{3000 - 2780}{900/\sqrt{36}} < Z < \frac{3500 - 2780}{900/\sqrt{36}}\right) \\ &= P(1.47 < Z < 4.8) \\ &= 0.0708 \end{aligned}$$

Example 2

On April 2nd, in 2007, Google sent an email to its employees warning that a python was loose in its New York Office. It was not a joke. The Burmese python is one of the largest species of snakes in the world. The lengths of these pythons are normally distributed with an average of 3.7 metres with a standard deviation of 0.3 meters.

a. At a zoo, there are 10 Burmese pythons on display. What is the probability that the average length of these pythons exceed 4 meters?

b. Your neighbour keeps two Burmese pythons in his basement. What is the probability that the average length of your neighbour's pets is less than 3.8 meters?

c. In the wild, eight Burmese pythons have been captured. What is the probability that these pythons have an average length between 3.6 and 3.9 meters?

d. What is the probability that a randomly selected Burmese python will be longer than 4 meters?

Solution

Let $\overline{X}$ = the average length of the pythons in the sample.

CLT: $\overline{X} \sim N\left(\mu, \frac{\sigma}{\sqrt{n}}\right) = N\left(3.7, \frac{0.3}{\sqrt{n}}\right)$

a. $P(\overline{X} > 4)$ for $n = 10$ Burmese pythons

   $P(\overline{X} > 4) = P(Z > 3.16) = 0.0008$

b. $P(\overline{X} < 3.8)$ for $n = 2$ Burmese pythons.

   $P(\overline{X} < 3.8) = P(Z < 0.47) = 0.6808$

c. $P(3.6 < \overline{X} < 3.9)$ for $n = 8$ Burmese pythons.

$$P\left(3.6 < \overline{X} < 3.9\right) = P(-0.94 < Z < 1.89) = 0.7963$$

d. Let X =the length of a Burmese python in meters.

$$X \sim N(\mu, \sigma) = N(3.7, 0.3)$$

$$P(X > 4) = P(Z > 1) = 0.1587$$

Example 3

The Heart Attack Grill in Las Vegas is famous for having a menu that consists only of unhealthy items. It makes a point of only serving food that is high in fat, sugar, and cholesterol. Patrons over 350 pounds eat for free, and the only vegan option on the menu is cigarettes. The average dinner bill per person at the Heart Attack Grill is \$164.68 with a standard deviation of \$14.44

a. Last Friday, 49 people had dinner at the Grill. What is the probability that the average of their bills was between \$160 and \$170?

b. Last Saturday, 64 ate dinner at the Grill. What is the probability that the average of their bills was more than \$161.50?

c. On Sunday, only 36 people came into the Grill for dinner. What is the probability that the average of their bills was under 172.68?

d. In parts (a)-(c), is it necessary to assume that the average amount spent at the grill is normally distributed? Why or why not? Explain with one or two sentences.

Solution

Let $\overline{X}$ = the mean bill for the patrons in the sample.

$$\overline{X} \sim N\left(\mu, \frac{\sigma}{\sqrt{n}}\right) = \left(164.68, \frac{14.44}{\sqrt{n}}\right)$$

a. $P(160 < \overline{X} < 170)$ for $n = 49$

$$P\left(160 < \overline{X} < 170\right) = P(-2.27 < Z < 2.58) = 0.9835$$

b. $P(\overline{X} > 161.50)$ for $n = 64$

$$P\left(\overline{X} > 161.50\right) = P(Z > -1.76) = 0.9608$$

c. $P(\overline{X} < 170.68)$ for $n = 36$

$$P\left(\overline{X} < 172.48\right) = P(Z < 3.24) = 0.9994$$

d. No, because all sample sizes are greater than 30. So the CLT will guarantee that the means will be normally distributed.

Example 4

Visitors to the Smithsonian National Air and Space Museum, can now enjoy listening to an audioguide that describes the museum's exhibits and offerings in Klingon. The time it takes for the average person to be fluent in Klingon, is normally distributed with a mean of 360 days and a standard deviation of 25 days. At school offering Klingon 101, there are 20 students registered in the course.

a. What is the probability that a randomly selected student will take less than 390 days to become fluent in Klingon?

b. If 10 students are selected from this class. What is the probability at least eight of them will take less than 390 days to become fluent in Klingon?

c. If 10 students are selected from this class, what is the probability that the average number of days for them to become fluent in Klingon is less than 390 days?

Solution

a. Let X = the number of days that is required to become fluent in Klingon.

$X \sim N(\mu, \sigma) = N(360, 25)$

$$P(X < 390) = P(Z < 1.20) = 0.8849$$

b. Let Y = the number of students who are fluent in Klingon in under 300 days.

$Y \sim \text{Binomial}(10, 0.8849)$

$$\begin{aligned} P(Y \geq 8) &= P(Y = 8) + P(Y = 9) + P(Y = 10) \\ &= C_8^{10}(0.8849)^8(0.1151)^2 + C_9^{10}(0.8849)^9(0.1151) \\ &\qquad + C_{10}^{10}(0.8849)^{10}(0.1151)^0 \\ &= 0.2241 + 0.3829 + 0.2944 \\ &= 0.9009 \end{aligned}$$

c. Let $\overline{X}$ = the average number of days required by the number of students in the sample to become fluent in Klingon.

$\overline{X} \sim N\left(\mu, \dfrac{\sigma}{\sqrt{n}}\right) = \left(360, \dfrac{25}{\sqrt{10}}\right)$

$P(\overline{X} < 390) = P(Z < 3.79) = 1$

Example 5

Suppose that you have a sample of 100 values from a population with mean $\mu = 500$ and with standard deviation $\sigma = 80$.

a. What is the probability that the sample mean will be in the interval $(490, 510)$?

b. Give an interval that covers the middle 95% of the distribution of the sample mean.

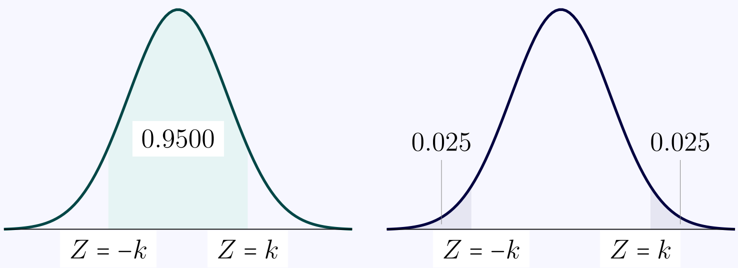
Solution

a. Let $\overline{X}$ = the sample mean.

Then $\overline{X} \sim N\left(\mu, \frac{\sigma}{\sqrt{n}}\right) = N\left(500, \frac{80}{\sqrt{100}}\right)$

$$
\begin{aligned}
P\left(490 < \overline{X} < 510\right) &= P(-1.25 < Z < 1.25) \\
&= 0.8944 - 0.1056 \\
&= 0.7888
\end{aligned}
$$

b. Want to find the values of k such that $P(-k < Z < k) = 0.95$



The closest value to 0.0250 in the table occurs when $Z = -1.96$. And since the graph is symmetric, this implies that $P(-1.96 < Z < +1.96) = 0.95$.

$$
\begin{aligned}
\overline{X} &= \mu \pm Z \cdot \frac{\sigma}{\sqrt{n}} \\
&= 500 \pm 1.96 \cdot \frac{80}{\sqrt{100}} \\
&= 500 \pm 15.68
\end{aligned}
$$

Therefore, $P(484.32 < \overline{X} < 515.68) = 0.95$