# L13. Confidence Interval on the Mean; Single Population, Variance Known

**Example 1**

Past experience has indicated that the breaking strength of yarn used in manufacturing drapery material is normally distributed and that $\sigma = 2$ psi. A random sample of nine specimens is tested, and the average breaking strength is found to be 98 psi.

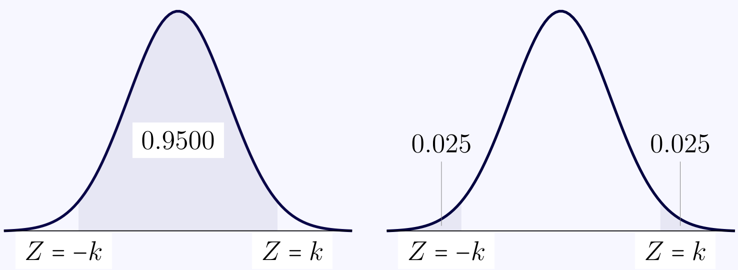
Find a 95% two-sided confidence interval on the true mean breaking strength.

**Solution**

We are given the following information:

$$n = 9 \qquad \overline{x} = 98 \qquad \sigma = 2$$

and want to construct a 95% confidence interval for the population mean $\mu$. This means that we have to find the $Z$−value which satisfies $P(-k < Z < k) = 0.95$




The critical value for a 95% confidence interval is: $Z_{\alpha/2} = Z_{0.025} = 1.96$

The confidence interval for $\mu$ therefore:

$$\overline{x} \pm Z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$$

$$98 \pm 1.96\frac{2}{\sqrt{9}}$$

$$98 \pm 1.3067 \quad \Rightarrow \quad (96.9633, 99.0307) \quad \Rightarrow \quad 96.9633 < \mu < 99.0307$$

**Interpretation:**

With repeated sampling, we are 95% confident that the true mean breaking strength of the yarn is between 96.9633 and 99.0307 psi.

**Note:** Even though the sample size is small, $n = 9 < 30$, the CLT still applies, because the mean breaking strength is normally distributed.

### Example 2

The diameter of a cable is known to have a normal distribution with $\sigma = 0.04$. A random sample of 25 cables was taken, and the average diameter for this sample was found to be 1.50 inches.

a. Find a 80% two-sided confidence interval for the mean diameter.

b. Find a 90% two-sided confidence interval on the mean diameter of the cable.

c. Find a 99% two-sided confidence interval on the mean diameter.

d. Which of the above intervals is the longest?

e. Which of the above intervals is the most precise?

### Solution

We are given

$$n = 25 \qquad \overline{x} = 1.5 \qquad \sigma = 0.04$$

a. 80% confidence interval $\Rightarrow$ $Z_{\alpha/2} = Z_{0.10} = 1.28$

$$\overline{x} \pm Z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}}$$
$$1.50 \pm 1.28 \cdot \frac{0.04}{\sqrt{25}}$$
$$1.50 \pm 0.0102 \quad \Rightarrow \quad (1.4898, 1.5102) \quad \Rightarrow \quad 1.4898 < \mu < 1.5102$$

**Interpretation:**

With repeated sampling, we are 80% confident that the true average diameter of the cable is between 1.4898 and 1.5102 inches.

b. 90% confidence interval $\Rightarrow$ $Z_{\alpha/2} = Z_{0.05} = 1.645$

$$\overline{x} \pm Z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}}$$
$$1.50 \pm 1.645 \cdot \frac{0.04}{\sqrt{25}}$$
$$1.50 \pm 0.0132 \quad \Rightarrow \quad (1.4868, 1.5132) \quad \Rightarrow \quad 1.4868 < \mu < 1.5132$$

**Interpretation:**

With repeated sampling, we are 90% confident that the true average diameter of the cable is between 1.4868 and 1.5132 inches.

c. 99% confidence interval $\Rightarrow$ $Z_{\alpha/2} = Z_{0.005} = 1.645$

$$\overline{x} \pm Z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}}$$
$$1.50 \pm 2.575 \cdot \frac{0.04}{\sqrt{25}}$$
$$1.50 \pm 0.0206 \quad \Rightarrow \quad (1.4794, 1.5206) \quad \Rightarrow \quad 1.4794 < \mu < 1.5206$$

**Interpretation:**

With repeated sampling, we are 99% confident that the true average diameter of the cable is between 1.4794 and 1.5206 inches.

d. The longest interval amongst the three, is the 99% confidence interval.

e. The most precise interval of the three is the 80% confidence interval.

### Example 3

The lifetime in hours of a 75-watt light bulb is known to be normally distributed with $\sigma = 25$ hours. A random sample of 20 bulbs has a mean life of $\overline{x} = 1014$ hours.

a. Construct a 95% two-sided confidence interval on the mean life.

b. Suppose that we want the error for estimating the mean life to be at most five hours at 95% confidence. What sample size should be used?

### Solution

We are given

$$n = 20 \qquad \overline{x} = 1014 \qquad \sigma = 25$$

a. 95% confidence interval $\Rightarrow$ $Z_{\alpha/2} = Z_{0.025} = 1.95$

$$\overline{x} \pm Z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}}$$

$$1014 \pm 1.96 \cdot \frac{25}{\sqrt{20}}$$

$$1014 \pm 10.9567 \quad \Rightarrow \quad (1003.0433, 1024.9567)$$

$$\Rightarrow \quad 1003.0433 < \mu < 1024.9567$$

**Interpretation:**

With repeated sampling, we are 95% confident that true average lifetime of the light bulbs is between 1003.0433 and 1024.9567 hours.

b. Want 95% confidence with maximum error, $E = 5$ hours, the sample size is

$$n = \left(\frac{Z_{\alpha/a}\sigma}{E}\right)^2 = \left(\frac{1.96 \cdot 25}{5}\right) = 96.04$$

$\Rightarrow$ 96 light bulbs are required.

**Example 4**

A bank manager wants to know the mean amount paid per month by home owners living in the town of Saint-Augustin in Mirabel. A random sample of 50 residents selected from the area showed that they pay an average of \$1575 per month for their mortgages. The population standard deviation for such mortgages is \$215.

a. Find a 97% confidence interval for the mean amount paid per month by all home owners in this area.

b. How large of a sample should be selected so that the estimate for the average monthly mortgage payment is within \$50 of the true population mean with 97% confidence?

c. How large of a sample should be selected so that the estimate for the average monthly mortgage payment is within \$50 of the true population mean with 95% confidence?

**Solution**

We have: $n = 50$ $\overline{x} = 1575$ $\sigma = 215$

a. 97% confidence interval $\Rightarrow$ $Z_{\alpha/2} = Z_{0.015} = 2.17$

$$\overline{x} \pm Z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}}$$
$$1575 \pm 2.17 \cdot \frac{215}{\sqrt{50}}$$
$$1575 \pm 65.9801 \quad \Rightarrow \quad (1509.012, 1640.9801)$$
$$\Rightarrow \quad 1509.01 < \mu < 1640.98$$

**Interpretation:**

With repeated sampling, we are 97% confident that true average mortgage payment in the town of Saint-Augustin is between \$ 1509.01 and \$ 1640.98

b. Want 97% confidence with $E = 50$

$$n = \left(\frac{Z_{\alpha/2} \cdot \sigma}{E}\right)^2 = \left(\frac{2.17 \cdot 215}{50}\right)^2 = 87.0676$$

$\Rightarrow$ 87 households need to be selected.

c. Want 95% confidence with $E = 50$

$$n = \left(\frac{Z_{\alpha/2} \cdot \sigma}{E}\right)^2 = \left(\frac{1.96 \cdot 215}{50}\right)^2 = 71.0312$$

$\Rightarrow$ 71 households need to be selected.

**Example 5**

Suppose that for a particular brand of concrete, the compressive strength of the material is known to be normally distributed with a variance of 6.25 psi. In a quality control test, 15 samples were taken, and their mean compressive strength was found to be 54.41 psi.

a. Determine a two-sided 98% confidence interval for the average compressive strength of the concrete.

b. Determine a 98% lower confidence bound for the true average compressive strength of the concrete.

**Solution**

We have: $n = 15 \qquad \overline{x} = 54.41 \qquad \sigma^2 = 6.25$

a. 98% confidence interval $\Rightarrow \quad Z_{\alpha/2} = Z_{0.01} = 2.33$

$$\overline{x} \pm Z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}}$$

$$54.41 \pm 2.33 \cdot \frac{\sqrt{6.25}}{\sqrt{15}}$$

$$54.41 \pm 1.504 \quad \Rightarrow \quad (52.906, 55.914)$$

$$\Rightarrow \quad 52.906 < \mu < 55.914$$

**Interpretation:**

With repeated sampling, we are 98% confident that true average compressive strength of the concrete is between 52.906 and 55.914 psi.

b. Want 98% lower confidence bound $\Rightarrow \quad Z_{\alpha} = Z_{0.02} = -2.05$

$$\begin{aligned} \overline{x} - Z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}} &\leq \mu \\ 54.41 - 2.05 \cdot \frac{\sqrt{6.25}}{\sqrt{15}} &\leq \mu \\ 53.0867 &\leq \mu \end{aligned}$$

**Interpretation:**

With repeated sampling, we are 98% confident that true average compressive strength of the concrete is at least 53.0867 psi.

**Example 6**

A city planner wants to estimate the average monthly water usage in the city of Montreal. He selects a random sample of 40 households, which gave a mean water usage to be 3415.70 gallons over a 1-month period. Suppose that it is known that water usage in Montreal follows a normal distribution with a standard deviation is 389.60 gallons.

a. Make an 85% confidence interval for the average monthly water usage for all households in Montreal.

b. What sample size is needed so that the estimate for the average monthly water usage in this city is within 60 gallons of the actual population mean with 85% confidence?

c. Construct an 85% upper-confidence bound for the mean water usage for all households in Montreal.

**Solution**

We are given: $n = 40$ $\quad \overline{x} = 3415.70$ $\quad \sigma = 389.60$

a. 85% confidence interval $\Rightarrow$ $Z_{\alpha/2} = Z_{0.075} = 1.44$

$$\begin{aligned} &\overline{x} \pm Z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}} \\ &3415.70 \pm 1.44 \cdot \frac{389.60}{\sqrt{40}} \\ &3415.70 \pm 88.7057 \quad \Rightarrow \quad (3326.9943, 3504.4057) \end{aligned}$$

$$\Rightarrow \quad 3326.9943 < \mu < 3504.4057$$

**Interpretation:**

With repeated sampling, we are 85% confident that true average water consumption in the city of Montreal is between 3326.9943 and 3504.4507 gallons/month.

b. 85% confidence with $E = 60$

$$n = \left( \frac{Z_{\alpha/2} \cdot \sigma}{\sqrt{n}} \right)^2 = \left( \frac{1.44 \cdot 389.60}{60} \right)^2 = 87.42$$

$\Rightarrow$ 87 households are needed.

c. 85% upper confidence interval $\Rightarrow$ $Z_\alpha = Z_{0.15} = 1.04$

$$\begin{aligned} \mu &\leq \overline{x} + Z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}} \\ \mu &\leq 3415.70 + 1.04 \cdot \frac{389.60}{\sqrt{40}} \\ \mu &\leq 3479.7652 \end{aligned}$$

**Interpretation:**

With repeated sampling, we are 85% confident that true average water consumption in the city of Montreal is at most 3479.7652 gallons/month.