L5. Probability Theory - Combinatorics

The Multiplication Rule

Example 1: Sandwiches

In 1896, New York passed a law that alcohol could only be served on Sunday if it was with a meal. New York taverns then started "selling" inedible sandwiches (served with a drink). The waiter would collect the sandwich at the end of the meal, and serve it the next customer.

At a shop in Manhattan, a customer can choose from the following ingredients to build their sandwich: six types of bread; eight types of protein; five types of cheese; and six condiments.

a. If a customer builds a sandwich by choosing one bread, one protein, one cheese, and one sauce, how many possible sandwiches can be made?"

b. The shop decides to add vegetables (tomatoes, lettuce, onions, etc.) to the list of choices. There are 10 vegetables available, and the customer must decide "yes" or "no" for each. With these new options, how many possible sandwiches can now be created?

Solution

a. Let S be the set of all sandwiches made by choosing exactly one bread, one protein, one cheese, and one condiment. By the multiplication rule,

$$n(S) = (6 \text{ breads}) \times (8 \text{ proteins}) \times (5 \text{ cheeses}) \times (6 \text{ condiments}) = 6 \cdot 8 \cdot 5 \cdot 6 = 1440.$$

b. Let V denote the vegetable choices. With 10 vegetables, each independently chosen "yes/no,"

$$n(V) = 2^{10} = 1024.$$

The total number of sandwiches with vegetables available is

$$n(S \text{ with veg}) = n(S) \times n(V) = 1440 \times 1024 = 1{,}474{,}560.$$

Example 2: Phone Number

While filming the 1996 horror movie Scream, a phone used by Drew Barrymore's character to make panicked 911 calls was accidentally left connected. Concerned police phoned back to check the murders weren't real.

a. If digits are allowed to repeat, how many 7-digit phone numbers are possible?

b. If digits are allowed to repeat, what is the probability that a 7-digit phone number is divisible by 2?

c. If digits are allowed to repeat, what is the probability that a 7-digit phone number is ends in 911?

d. If no repetition are allowed, how many 7-digit phones numbers are there?

e. Assuming that digits are not allowed to repeat, what is the probability that a randomly selected 7-digit phone number is divisible by 5?

Solution

Assumption: A 7-digit phone number is a string of 7 digits from $\{0, 1, \ldots, 9\}$. Leading zeros are allowed unless otherwise stated.

a. Let S be the set of all 7-digit strings (digits may repeat).

$$n(S) = 10^7 = 10{,}000{,}000.$$

b. Let E be "number is divisible by 2." With repetition allowed, the last digit must be even ($0, 2, 4, 6, 8$ — 5 options), and the first six digits are arbitrary:

$$n(E) = 10^6 \cdot 5 \quad \Rightarrow \quad P(E) = \frac{10^6 \cdot 5}{10^7} = \frac{1}{2} = 0.5.$$

c. Let C be "number ends with 911." The first four digits are arbitrary (10^4 options), and the last three are fixed as 9–1–1:

$$n(C) = 10^4 \quad \Rightarrow \quad P(C) = \frac{10^4}{10^7} = 10^{-3} = 0.001.$$

d. If no repetitions are allowed, the count is a permutation of length 7 from 10 digits:

$$n = {}_{10}P_7 = 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 = 604{,}800.$$

e. With no repetition, let F be "number is divisible by 5." The last digit must be 0 or 5 (2 choices). For each choice of last digit, the remaining 6 positions are filled by a permutation of 6 distinct digits chosen from the remaining 9 digits:

$$n(F) = 2 \cdot {}_9P_6 = 2 \cdot (9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4) = 120{,}960.$$

Hence

$$P(F) = \frac{n(F)}{{}_{10}P_7} = \frac{120{,}960}{604{,}800} = \frac{1}{5} = 0.2.$$

Example 3: Vacation Packages

n 1988, Yellow Pages had to pay \$18 million to a travel agency after it accidental advertised them as a provider of 'erotic holidays' instead of 'exotic holidays'.

A travel company now offers customizable vacation packages. Each customer can choose:

- 3 destinations (beach, city, or mountains)
- 4 hotel types (budget, standard, deluxe, luxury)
- 2 meal plans (all-inclusive or breakfast-only)
- 5 optional excursions (each may be chosen or not chosen independently)

a. If a customer must pick exactly one destination, one hotel, and one meal plan, but no excursions, how many vacation packages are possible?

b. If the excursions are also available, and the customer must decide "yes" or "no" for each of the 5 excursions, how many possible vacation packages can now be created?

c. Suppose the customer wants to take *at least one excursion*. How many vacation packages does this allow?

d. If the company introduces 3 flight classes (economy, premium economy, business), and every customer must choose one, how many total vacation packages are possible now (including excursions)?

e. Suppose a customer insists on staying in a luxury hotel and choosing the all-inclusive meal plan. With those restrictions, how many possible vacation packages can be built (including excursions)?

Solution

a. Let S_0 be the set of packages with one destination, one hotel, one meal plan, and no excursions.

$$n(S_0) = (3 \text{ destinations}) \cdot (4 \text{ hotels}) \cdot (2 \text{ meals}) = 3 \cdot 4 \cdot 2 = 24.$$

b. With 5 optional excursions, each independently "yes/no," there are $2^5 = 32$ excursion patterns. Thus

$$n(S) = n(S_0) \cdot 2^5 = 24 \cdot 32 = 768.$$

c. "At least one excursion" excludes the "no excursion" pattern. There are $2^5 - 1 = 31$ nonempty excursion choices, so

$$n(\text{at least one}) = n(S_0) \cdot (2^5 - 1) = 24 \cdot 31 = 744.$$

d. Add 3 flight classes (must choose one). Multiply the count from (b) by 3:

$$n(\text{with flights}) = 3 \cdot 768 = 2304.$$

e. Restrict to luxury hotel (1 option) and all-inclusive meal (1 option). Keep 3 destinations and all excursion patterns:

$$n(\text{luxury \& all-inc}) = (3)\cdot(1)\cdot(1)\cdot 2^5 = 3\cdot 32 = 96.$$

Example 4: Passwords

In what may be the most expensive "Forgot Password" moment in history, Stefan Thomas is locked out of his Bitcoin wallet containing \$ 220 million. He has two guesses left before it permanently locks him out.

A computer system uses passwords that are six characters long, and each character is one of 26 lower case letters (`a, z`), 26 upper case letters (`A, Z`), and 10 integers (`0, 9`).

a. How many six character passwords are possible?

b. What is the probability that a randomly selected password contains only letters? (Repetition allowed)

c. What is the probability that a randomly selected password contains only letters? (Repetition not allowed).

d. What is the probability that a randomly selected password spells the word "cat" (using upper and lower case letters), followed by a string of 3-digits (repetition allowed on the numbers)?

e. What is the probability that password contains exactly one digit and five letters? (Repetition allowed).

Solution

a. Total number of passwords

$$n(S) = 62^6 = 5.68\times 10^{10}$$

b. Let OL = the password contains only letters

$$P(OL) = \frac{n(OL)}{n(S)} = \frac{52^6}{62^6} = 0.3481$$

c. Let $OLNR$ = the password contains only letters, no repetition

$$P(OLNR) = \frac{n(OLNR)}{n(S)} = \frac{52\cdot 51\cdot 50\cdots 47}{62^6} = 0.2581$$

d. Let C = the password starts with *cat*

$$P(C) = \frac{n(C)}{n(S)} = \frac{2\cdot 2\cdot 2\cdot 10\cdot 10\cdots 10}{62^6} = 1.4084\times 10^7$$

e. Let D = the password contains exactly 1 digit

$$P(D) = \frac{n(D)}{n(S)} = \frac{C_1^6 \cdot 10 \cdot 52^5}{62^6} = 0.4016$$

Combinations

Example 5: Yellow Pages

Due to an embarrassing set of initials, in 2009 the Wisconsin Tourism Federation changed its name to the Tourism Federation of Wisconsin.

To promote travel within the state, the Federation offers $n = 12$ distinct attractions (e.g., breweries, dairy farms, hiking trails, cheese factories, state parks, art festivals, etc.). Visitors can create their own tour package by choosing from these attractions.

a. How many ways can a visitor choose all 12 attractions?

b. How many ways can a visitor choose exactly 4 attractions for a weekend trip?

c. How many ways can a visitor choose exactly 6 attractions for a week-long trip, if order does not matter?

d. Suppose the visitor wants to split their day into 2 morning attractions and 3 afternoon attractions. How many different itineraries are possible?

e. The Federation launches a "Taste of Wisconsin" pass where visitors must pick 1 brewery tour, 2 nature excursions, and 2 cultural events. If there are 5 brewery tours, 4 nature excursions, and 3 cultural events available, how many different passes can be created?

Solution

a. Choosing all 12 from 12:

$$_{12}C_{12} = 1.$$

b. Exactly 4 attractions:

$$_{12}C_4 = \frac{12!}{4!\,8!} = 495.$$

c. Exactly 6 attractions (order does not matter):

$$_{12}C_6 = \frac{12!}{6!\,6!} = 924.$$

d. Split into 2 morning and 3 afternoon (disjoint choices):

$$_{12}C_2 \times_{10} C_3 = 66 \times 120 = 7920.$$

e. "Taste of Wisconsin" pass:

$$_5C_1 \times_4 C_2 \times_3 C_2 = 5 \times 6 \times 3 = 90.$$

Example 6: Coin Toss

In Pirates of the Caribbean: Dead Man's Chest (2006), Will Turner and his crew must decide who will board the cursed Flying Dutchman. To make it fair, they settle it the pirate way—by flipping a coin.

But what if, instead of just one toss, they decide to get really dramatic and flip 10 coins to determine fate?

Each pirate flips a fair coin (Heads = Goes aboard, Tails = Stays safe).

a. What is the probability that exactly 5 pirates end up aboard the Flying Dutchman?

b. What is the probability that at least 8 pirates are forced onto the Dutchman?

c. Will Turner only boards the Flying Dutchman if he flips 10 heads in a row. What's the probability that he meets his watery doom?

d. If a cursed coin increases the chance of landing heads to 60%, how does this affect the probability that at two or more of the crewmen are sent aboard?

Solution

Model: Interpret the drama as $n = 10$ independent flips (or 10 pirates each flipping once). Let X =count the number of Heads (aboard).

a. Exactly 5 aboard with a fair coin ($p = \frac{1}{2}$):

$$P(X = 5) = {}_{10}C_5 \left(\tfrac{1}{2}\right)^5 \left(\tfrac{1}{2}\right)^5 = \frac{{}_{10}C_5}{2^{10}} = \frac{252}{1024} \approx 0.2461.$$

b. At least 8 aboard (fair coin):

$$P(X \geq 8) = \frac{{}_{10}C_8 + {}_{10}C_9 + {}^{10}C_{10}}{2^{10}} = \frac{45 + 10 + 1}{1024} = \frac{56}{1024} \approx 0.0547.$$

c. Will boards only if he gets 10 Heads in a row (fair coin):

$$P(10 \text{ Heads}) = \left(\tfrac{1}{2}\right)^{10} = \frac{1}{1024} \approx 0.0010.$$

d. Let $X \sim \text{Binomial}(n = 10, p = 0.60)$ count the number of Heads (aboard). We want $P(X \geq 2) = 1 - P(X = 0) - P(X = 1)$.

$$\begin{aligned} P(X \geq 2) &= 1 - \left[{}^{10}C_0 (0.60)^0 (0.40)^{10} + {}^{10}C_1 (0.60)^1 (0.40)^9 \right] \\ &= 1 - \left[(0.40)^{10} + 10(0.60)(0.40)^9 \right] \\ &= 1 - 0.0016777216 \\ &\approx 0.9983. \end{aligned}$$

Example 7: Bombay Gin

Last year, Bombay Sapphire gin was recalled after some bottles were found to contain 77% alcohol instead of the usual 40%. A Bacardi spokesperson assured the public it was "not unsafe to drink" — it was just not recommended. Despite the recall, only a fifth of the bottles were returned.

You're at the store, staring at 20 bottles, 12 of which are the extra-boozy batch. You grab four at random and toss them into your cart.

a. exactly two of the bottles in your shopping cart contain more alcohol than usual?

b. more than half of the bottles in your trolley contain more alcohol than usual?

Solution

Model: This is sampling *without replacement* from a finite population,
Let X = the number of extra-boozy bottles in the cart

$$N = 20 \text{ (total bottles)}, \quad K = 12 \text{ (extra-boozy)}, \quad n = 4 \text{ (draws)}$$

a. Exactly two extra-boozy:

$$P(X = 2) = \frac{{}_{12}C_2 \, {}_8C_2}{{}^{20}C_4} = \frac{66 \cdot 28}{4845} = \frac{616}{1615} \approx 0.3814.$$

b. More than half of the cart (i.e., at least 3 of 4):

$$P(X \geq 3) = P(X = 3) + P(X = 4) = \frac{{}_{12}C_3 \, {}_8C_1 + {}_{12}C_4 \, {}_8C_0}{{}^{20}C_4} = \frac{220 \cdot 8 + 495 \cdot 1}{4845} = \frac{451}{969} \approx 0.4654.$$

Example 8: See you in the Bahamas

For 30 years, one man, Donald Lau, wrote every fortune and lucky number for Wonton Food Inc., the world's largest fortune cookie supplier. Then, in 2005, he accidentally predicted the lottery. His fortune read: *"All the preparation you've done will finally be paying off,"* followed by the numbers 22, 28, 32, 33, 39, 40.

It turned out 110 people took that very literally, winning second place in the New York Lotto and triggering a fraud investigation. Turns out, some fortunes really do come true —just not for the guy who wrote them.

a. How many ways can six numbers be chosen from 49?

b. Suppose that you picked six numbers. What is the probability that

   i. none of your numbers match the winning six?

   ii. one of your numbers match the winning six?

   iii. two of your numbers match the winning six?

iv. three of your numbers match the winning six?
v. four of your numbers match the winning six?
vi. five of your numbers match the winning six?
vii. all six of your numbers match the winning six?

Solution

Model: A Lotto draw selects 6 distinct numbers from $\{1, \ldots, 49\}$. Let X be the number of matches between your chosen 6 numbers and the (fixed) winning 6.

a. Total ways to choose the winning set:

$$_{49}C_6 = 13{,}983{,}816.$$

b. For a fixed winning set, and a random ticket of 6 numbers:

i. None match

$$P(X = 0) = \frac{_6C_0\ _{43}C_6}{_{49}C_6} = \frac{1 \cdot 6{,}096{,}454}{13{,}983{,}816} \approx 0.43597.$$

ii. Exactly one matches

$$P(X = 1) = \frac{_6C_1\ _{43}C_5}{_{49}C_6} = \frac{5{,}775{,}588}{13{,}983{,}816} \approx 0.41302.$$

iii. Exactly two match (Free play)

$$P(X = 2) = \frac{_6C_2\ _{43}C_4}{_{49}C_6} = \frac{1{,}851{,}150}{13{,}983{,}816} \approx 0.13238.$$

iv. Exactly three match ($ 10)

$$P(X = 3) = \frac{_6C_3\ _{43}C_3}{49C_6} = \frac{246{,}820}{13{,}983{,}816} \approx 0.01765.$$

v. Exactly four match (Share of 4% of the Pool's Fund)

$$P(X = 4) = \frac{_6C_4\ _{43}C_2}{_{49}C_6} = \frac{15 \cdot 903}{13{,}983{,}816} \approx 0.0009686.$$

vi. Exactly five match (Share of 5% of the Pool's Fund)

$$P(X = 5) = \frac{_6C_5\ _{43}C_1}{_{49}C_6} = \frac{258}{13{,}983{,}816} \approx 1.845 \times 10^{-5}.$$

vii. All six match (Jackpot win or Share of 79.5% of the Pool's Fund)

$$P(X = 6) = \frac{_6C_6\ _{43}C_0}{_{49}C_6} = \frac{1 \cdot 1}{13{,}983{,}816} \approx 7.151 \times 10^{-8}.$$

Example 9: PokerTox

Surgeons in New York are now offering Pokertox - botox injections that neutralize any ticks or tells on the face that may give away a good/bad hand during a game of cards. The procedure costs \$600 – \$800, and needs to be repeated once every three to four weeks.

Definition:

$$\text{rank} = \{A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K\}$$
$$\text{suit} = \{\clubsuit, \heartsuit, \spadesuit, \diamondsuit\}$$

a. How many 5 card poker hands can be distributed from a deck of 52 cards?

b. What is the probability of being dealt a single pair?

c. What is the probability of being dealt a three-of-kind?

d. What is the probability of being dealt a four-of-a-kind?

Solution

a. Total number of poker hands

$$n(S) = {}_{52}C_5 = 2598960$$

b. Let Pa = being dealt a pair

$$P(Pa) = \frac{n(Pa)}{n(S)} = \frac{{}_{13}C_1\ {}_4C_2\ {}_{12}C_3\ {}_4C_1\ {}_4C_1\ {}_4C_1}{{}_{52}C_5} = \frac{1098240}{2598960} = 0.4226$$

c. Let T = being dealt a three of a kind

$$P(T) = \frac{n(T)}{n(S)} = \frac{{}_{13}C_1\ {}_4C_3\ {}_{12}C_2\ {}_4C_1\ {}_4C_1}{{}_{52}C_5} = \frac{54912}{2598960} = 0.0211$$

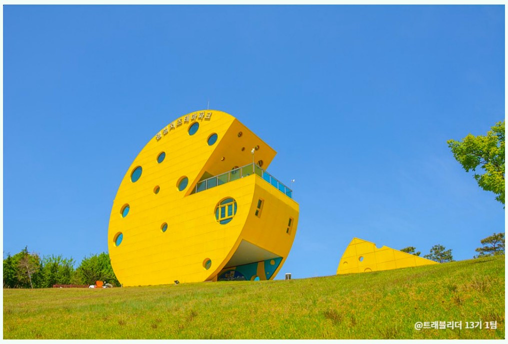
d. Let F = being dealt a four of a kind

$$P(F) = \frac{n(F)}{n(S)} = \frac{{}_{13}C_1\ {}_4C_4\ {}_{12}C_1\ {}_4C_1}{{}_{52}C_5} = \frac{624}{2598960} = 2.401 \times 10^{-4}$$

Permutations - Distinct Objects

Example 10: Cheese Park Lineups

There is a cheese-themed theme park in South Korea.



At one area of the park, there are 6 distinct cheese booths/attractions: *cheddar (C), gouda (G), brie (B), swiss (S), mozzarella (M), parmesan (P)*. Visitors plan the **order** in which they'll visit booths.

a. In how many ways can a visitor arrange the order to visit all 6 booths?

b. If a visitor chooses only 3 of the 6 booths, in how many different orders can they do so?

c. If the visitor must start at cheddar and end at parmesan, how many valid orders are there for visiting all 6 booths?

d. If cheddar and gouda must be *adjacent* (in either order), how many valid orders are possible for all 6 booths?

e. If cheddar and gouda must *not* be adjacent, how many valid orders are possible?

f. In how many orders is *brie before swiss* (not necessarily adjacent)?

g. If cheddar must be first and brie must appear somewhere before swiss, how many valid orders are there?

Solution

a. All 6 booths in some order:

$$6! = 720.$$

b. Choose and order 3 of the 6 booths (order matters):

$$_6P_3 = \frac{6!}{(6-3)!} = 6 \cdot 5 \cdot 4 = 120.$$

c. Fix first = C and last = P; arrange the remaining 4:

$$4! = 24.$$

d. C and G adjacent (either order). Treat (CG) or (GC) as one block: block + other 4 booths ⇒ 5! linear orders; internal 2 orders for the block:

$$2 \cdot 5! = 2 \cdot 120 = 240.$$

e. Not adjacent = total − adjacent:

$$6! - 240 = 720 - 240 = 480.$$

f. "B before S" (not necessarily adjacent). In all linear orders of distinct items, exactly half have B before S:

$$\frac{6!}{2} = \frac{720}{2} = 360.$$

g. Fix first = C. Arrange the remaining 5 booths; among these 5! orders, exactly half have B before S:

$$\frac{5!}{2} = \frac{120}{2} = 60.$$

Example 11: Race Podiums

At the end of the 17th Century, the Quaker town of Morley declared a rule against foot races, calling them "unfruitful works of darkness."

Now, centuries later, a local meet in Morley has 10 distinct runners competing for the podium (gold–silver–bronze). There are no ties.

a. How many distinct podium orders (gold–silver–bronze) are possible?

b. If runner A must win *gold*, how many podiums are possible?

c. If runner A must be on the podium (in any position), how many podiums are possible?

d. If both runners A and B must be on the podium (in any positions), how many podiums are possible?

e. Among the podiums where both A and B are on the podium, how many have A finishing ahead of B?

f. If A must *not* be on the podium, how many podiums are possible?

g. If neither Q nor R may appear on the podium, how many podiums are possible?

h. At least one of the three stars $\{A, B, C\}$ must be on the podium. How many podiums satisfy this?

Solution

a. Total podium orders (gold–silver–bronze) from 10 runners:

$$_{10}P_3 = \frac{10!}{7!} = 10 \cdot 9 \cdot 8 = 720.$$

b. Fix A as gold; choose and order silver/bronze from the remaining 9:

$$_9P_2 = 9 \cdot 8 = 72.$$

c. A must be on the podium (any position). Place A (3 choices), then order the other two from the remaining 9:

$$3 \cdot {}_9P_2 = 3 \cdot 9 \cdot 8 = 216.$$

d. Both A and B must be on the podium (any positions). Choose their two podium positions and order them (${}_3P_2 = 3 \cdot 2 = 6$ ways), then choose the third medalist from the remaining 8:

$$6 \cdot 8 = 48.$$

e. Among podiums with both A and B, exactly half have A ahead of B (by symmetry):

$$\frac{48}{2} = 24.$$

f. A must not be on the podium: choose and order 3 from the remaining 9:

$$_9P_3 = 9 \cdot 8 \cdot 7 = 504.$$

g. Neither Q nor R may appear: choose and order 3 from the remaining 8:

$$_8P_3 = 8 \cdot 7 \cdot 6 = 336.$$

h. At least one of $\{A, B, C\}$ is on the podium. Use complement:

$$\text{Total } - (\text{no } A, B, C) = {}_{10}P_3 - {}_7P_3 = 720 - (7 \cdot 6 \cdot 5) = 720 - 210 = 510.$$

Example 12: Concert Lineups

In 2016, a 16-year-old girl's emergency hospitalization was reported in the *Journal of Emergency Medicine* under the title "Boy Band–Induced Pneumothorax." She had a collapsed lung from screaming so loudly at a One Direction concert.

A festival has 7 distinct bands scheduled: *A, B, C, D, E, F, G*. They will perform in a single main-stage lineup (one after another), and order matters.

a. In how many ways can all 7 bands be ordered for the lineup?

b. If band A must be the *headliner* (last), how many valid orders are there?

c. If only 4 of the 7 bands are selected to perform, how many different *orders* can the show have?

d. If A and B must perform *consecutively* (in either order), how many valid full-lineup orders exist?

e. If A must open (first) and G must close (last), and C must perform somewhere *before* D, how many valid orders exist?

Solution

a. All 7 bands in some order:
$$7! = 5040.$$

b. Fix headliner = A; arrange the remaining 6:
$$6! = 720.$$

c. Order 4 of the 7 bands (partial permutation):
$${}_7P_4 = \frac{7!}{3!} = 7 \cdot 6 \cdot 5 \cdot 4 = 840.$$

d. A and B consecutive (either order). Treat (AB) or (BA) as one block: block + the other 5 bands $\Rightarrow$ 6! linear orders; internal 2 orders for the block:
$$2 \cdot 6! = 2 \cdot 720 = 1440.$$

e. Fix opener = A and closer = G. Arrange the 5 middle bands; among the 5! orders, exactly half have C before D:
$$\frac{5!}{2} = \frac{120}{2} = 60.$$

Permutations - Non Distinct/Repeated Objects

Example 13: Elements

Oxygen, nitrogen, and hydrogen are essential to life. Oxygen makes up most of the human body and fuels respiration. Nitrogen is the main ingredient in Earth's atmosphere and vital for DNA. Hydrogen is the lightest element and powers the Sun.

A scientist is organizing 7 labelled containers of these elements in a straight line: 2 oxygen; 3 nitrogen; and 2 hydrogen. Since identical containers are indistinguishable, the scientist wonders:

a. How many unique ways can the 7 containers be arranged?

b. If arranged randomly, what is the probability that the first two are oxygen samples?

c. If oxygen containers must stay together, how many unique sequences can be created?

Solution

a. Total number of arrangements

$$n(S) = \frac{7!}{2!3!2!} = 210$$

b. Let OO = the first two samples are oxygen

$$n(OO) = \frac{5!}{2!3!} = 10$$

c. Let O = Number of unique sequences where the oxygen's are together:

$$n(O) = \frac{5!}{2!3!} \times 6 \text{ spots} = \frac{6!}{2!3!} = 720$$

Example 14: Aluminum

Wikipedia once hosted a 40,000-word edit war over whether the metal should be spelled ”aluminum” or ”aluminium.” After relentless debates, countless edits, and possibly a few broken keyboards, the longer British spelling — ”aluminium” — won the battle.

a How many different arrangements can be made using all the letters in ”ALUMINUM”?

b How many different arrangements are possible if the sequence must begin with A and end with M?

c How many different arrangements are possible if the two M's must stay together as a single unit?

d What is the probability that a randomly selected arrangement of the letters has the M's together?

e What is the probability that the M's are not together?

Solution

a. Total number of arrangements

$$n(S) = \frac{8!}{2!2!} = 10080$$

b. Total number of arrangements that start with A and end with M

$$n(S) = \frac{6!}{2!} = 360$$

c. Total number of arrangements that have the M's together

$$n(M) = \frac{6!}{2!} \times 7 = \frac{7!}{2!} = 2520$$

d. Probability that the M's are together

$$P(M) = \frac{n(M)}{n(S)} = \frac{2520}{10080} = \frac{1}{4}$$

e. Probability that the M's are not together

$$P(M') = 1 - P(M) = 1 - \frac{1}{4} = \frac{3}{4}$$

Example 15: Books

Back in the Middle Ages, books were so valuable that they were literally chained to shelves to prevent theft. One of the few remaining examples is at Hereford Cathedral in England, where books are still locked up like they might make a run for it.

A librarian at the Grand Academy Library is reorganizing a shelf with 10 books. However, the books belong to different categories, and some are identical copies! The books are categorized as follows: 4 Math books; 3 Physics books; and 3 Literature books.

a. How many unique ways can the librarian arrange all 10 books on the shelf?

b. How many unique arrangements are possible if a Math book must always be placed first and last on the shelf?

c. How many different ways can the books be arranged if all Physics books must stay together as a single unit?

d. What is the probability that in a randomly arranged shelf, all Literature books are next to each other?

e. What is the probability that the Physics books are not next to each other?

Solution

a. Total number of arrangements

$$n(S) = \frac{10!}{4!3!3!} = 4200$$

b. Total number of arrangements that start with, and ends with math

$$n(S) = \frac{8!}{2!3!3!} = 560$$

c. Total number of arrangements if the physics books have to be together

$$n(P) = \frac{7!}{4!3!} \times 8 = 280$$

d. Let L = the literature books are together

$$P(L) = \frac{n(L)}{n(S)} = \frac{280}{4200} = \frac{1}{15}$$

e. Probability that the literature books are not together

$$P(L') = 1 - P(L) = 1 - \frac{1}{15} = \frac{14}{15}$$