L6. Rules of Probability

The Addition Rule

Example 1

One of the longest movies ever made, 'The Longest Most Meaningless Movie in the World', was released in 1968. It was two days long.

A survey of 500 people was conducted to determine their preferences for action and comedy movies. The results show that: 320 people like action movies; 280 people like comedy movies; and 150 people like both genres.

a. What is the probability that a randomly selected person likes either action or comedy movies?

b. What is the probability that a randomly selected person does not like either genre?

c. If one person is selected at random, what is the probability that they like only one of the two genres?

Solution

Let A = number of people who like action movies

C = the set of people who like comedy movies.

$$n(A) = 320, \quad n(C) = 280, \quad n(A \cap C) = 150, \quad n(S) = 500$$

a. What is the probability that a randomly selected person likes either action or comedy movies?

$$\begin{aligned} P(A \cup C) &= P(A) + P(C) - P(A \cup C) \\ &= \frac{320}{500} + \frac{280}{500} - \frac{150}{500} \\ &= \frac{450}{500} \\ &= \frac{9}{10} \end{aligned}$$

b. What is the probability that a randomly selected person does not like either genre?

$$\begin{aligned} P(\text{neither}) &= P(A \cup C)' \\ &1 - P(A \cup C) \\ &= 1 - \frac{9}{10} \\ &= \frac{1}{10} \end{aligned}$$

c. What is the probability that a randomly selected person likes only one of the two genres?

$$\begin{aligned} P(\text{only one}) &= P(A \cup C) - P(A \cap C) \\ &= \frac{450}{500} - \frac{150}{500} \\ &= \frac{300}{500} \\ &= \frac{3}{5} \end{aligned}$$

Example 2

In 1874, Max Planck's teacher told him not to go into physics, because the field was almost completely known and 'will arguably soon take its final form. Planck went on to make enormous contributions to quantum theory and won a Nobel Prize.

A professor analyzed the exam performance of 250 students in Mathematics and Physics. The results show that: 140 students passed Mathematics; 120 students passed Physics; and 60 students passed both subjects.

a. What is the probability that a randomly selected student passed at least one of the two subjects?

b. What is the probability that a randomly selected student failed both subjects?

c. What is the probability that a randomly selected student passed exactly one subject?

Solution

Let M = the set of students who passed Mathematics

P = the set of students who passed Physics.

$$n(M) = 140, \quad n(P) = 120, \quad n(M \cap P) = 60, \quad n(S) = 250$$

a. What is the probability that a randomly selected student passed at least one of the two subjects?

$$
\begin{aligned}
P(M \cup P) &= P(M) + P(P) - P(M \cap P) \\
&= \frac{140}{250} + \frac{120}{250} - \frac{60}{250} \\
&= \frac{200}{250} \\
&= \frac{4}{5}
\end{aligned}
$$

b. What is the probability that a randomly selected student failed both subjects?

$$
\begin{aligned}
P(\text{neither}) &= P(M \cup P)' \\
&= 1 - P(M \cup P) \\
&= 1 - \frac{4}{5} \\
&= \frac{1}{5}
\end{aligned}
$$

c. What is the probability that a randomly selected student passed exactly one subject?

$$
\begin{aligned}
P(\text{only one}) &= P(M \cup P) - P(M \cap P) \\
&= \frac{200}{250} - \frac{60}{250} \\
&= \frac{140}{250} \\
&= \frac{14}{25}
\end{aligned}
$$

Example 3

Hoping to contribute to science? The University of Nottingham is conducting a survey about what rabbits are cutest. Options for why you like a particular rabbit include ”it looks fluffy”, ”it looks soft”, and ”it looks like a rabbit”.

A biologist is studying a population of 300 rabbits to analyze two genetic traits: white fur (W) and long ears (L). The study finds that: 180 rabbits have the white fur trait; 140 rabbits have the long ears trait; 70 rabbits have both traits.

a. What is the probability that a randomly selected rabbit has at least one of the two traits (white fur or long ears)?

b. What is the probability that a randomly selected rabbit has neither of these traits?

c. What is the probability that a randomly selected rabbit has long ears only?

Solution

Let W = the set of rabbits with white fur

L = the set of rabbits with long ears.

$$n(W) = 180, \quad n(L) = 140, \quad n(W \cap L) = 70, \quad n(S) = 300$$

a. What is the probability that a randomly selected rabbit has at least one of the two traits (white fur or long ears)?

$$\begin{aligned} P(W \cup L) &= P(W) + P(L) - P(W \cap L) \\ &= \frac{180}{300} + \frac{140}{300} - \frac{70}{300} \\ &= \frac{250}{300} \\ &= \frac{5}{6} \end{aligned}$$

b. What is the probability that a randomly selected rabbit has neither of these traits?

$$\begin{aligned} P(\text{neither}) &= P(W \cup L)' \\ &= 1 - P(W \cup L) \\ &= 1 - \frac{5}{6} \\ &= \frac{1}{6} \end{aligned}$$

c. What is the probability that a randomly selected rabbit has long ears only?

$$\begin{aligned} P(\text{long ears only}) &= P(L) - P(W \cap L) \\ &= \frac{140}{300} - \frac{70}{300} \\ &= \frac{70}{300} \\ &= \frac{7}{30} \end{aligned}$$

Example 4

Flower petals have evolved to attract bees by reflecting a blue halo that is visible to bees but not humans. A botanist is studying the pollination of flowers by bees and butterflies in a garden of 400 flowers. The study finds that: 260 flowers were visited by bees; 190 flowers were visited by butterflies; 340 flowers were visited by at least one of the two types of pollinators (bees or butterflies).

a. Using the given information, calculate how many flowers were visited by both bees and butterflies.

b. What is the probability that a randomly chosen flower was visited by both bees and butterflies?

c. What is the probability that a randomly chosen flower was visited by only butterflies?

Solution

Let B = the set of flowers visited by bees

F the set of flowers visited by butterflies.

$$n(B) = 260, \quad n(F) = 190, \quad n(B \cup F) = 340, \quad n(S) = 400$$

a. Using the given information, calculate how many flowers were visited by both bees and butterflies.

$$\begin{aligned} n(B \cap F) &= n(B) + n(F) - n(B \cup F) \\ &= 260 + 190 - 340 \\ &= \boxed{110} \end{aligned}$$

b. What is the probability that a randomly chosen flower was visited by both bees and butterflies?

$$P(B \cap F) = \frac{n(B \cap F)}{n(S)} = \frac{110}{400} = \frac{11}{40}$$

c. What is the probability that a randomly chosen flower was visited by only butterflies?

$$\begin{aligned} P(\text{only butterflies}) &= P(F) - P(B \cap F) \\ &= \frac{190}{400} - \frac{110}{400} \\ &= \frac{80}{400} = \frac{1}{5} \end{aligned}$$

Conditional Probability

Example 5

A study of 600 adults investigates whether pet owners exercise regularly.

Exercise Habit	Owns a Pet	Does Not Own a Pet
Exercises Regularly	200	150
Does Not Exercise Regularly	180	70

a. If a randomly selected person owns a pet, what is the probability that they exercise regularly?

b. If a randomly selected person does not exercise regularly, what is the probability that they do not own a pet?

c. If a randomly selected person exercises regularly, what is the probability that they own a pet?

Solution

Let E = exercises regularly
P = owns a pet

Updated table with row and column totals:

Exercise Habit	P	P'	Total
E	200	150	350
E'	180	70	250
Total	380	220	600

a. We already know that they own a pet. So we want $P(E \mid P)$

$$P(E \mid P) = \frac{P(E \cap P)}{P(P)} = \frac{200/600}{380/600} = \frac{200}{380} = \frac{10}{19}$$

b. We already know that they do not exercise. So we want $P(P' \mid E')$

$$P(\bar{P} \mid \bar{E}) = \frac{P(\bar{P} \cap \bar{E})}{P(\bar{E})} = \frac{70/600}{250/600} = \frac{70}{250} = \frac{7}{25}$$

c. We already know that they exercise. So we want $P(P \mid E)$

$$P(P \mid E) = \frac{P(P \cap E)}{P(E)} = \frac{200/600}{350/600} = \frac{200}{350} = \frac{4}{7} \approx 0.571$$

Example 6

According to a study of 10 European countries, people in the UK have been eating more fruit and vegetables during the pandemic, but they also reported the largest increase in consumption of cereals, alcohol, comfort foods, and 'tasty treats'.

A supermarket tracks whether customers buy fruits or vegetables during a shopping trip. The data from 800 customers is recorded below.

Purchase Behavior	Bought Fruits	Did Not Buy Fruits
Bought Vegetables	300	200
Did Not Buy Vegetables	100	200

a. If a randomly selected customer bought vegetables, what is the probability that they also bought fruits?

b. If a randomly selected customer did not buy fruits, what is the probability that they also did not buy vegetables?

c. If a customer bought fruits, what is the probability that they did not buy vegetables?

Solution

Let F = bought fruits
V = bought vegetables

Updated table with row and column totals:

Purchase Behavior	F	F'	Total
V	300	200	500
V'	100	200	300
Total	400	400	800

a. We already know they bought vegetables. So we want $P(F \mid V)$

$$P(F \mid V) = \frac{P(F \cap V)}{P(V)} = \frac{300/800}{500/800} = \frac{300}{500} = \frac{3}{5}$$

b. We already know they did not buy fruits. So we want $P(V' \mid F')$

$$V' \mid F') = \frac{P(V' \cap F')}{P(F')} = \frac{200/800}{400/800} = \frac{200}{400} = \frac{1}{2}$$

c. We already know they bought fruits. So we want $P(V' \mid F)$

$$P(V' \mid F) = \frac{P(B' \cap F)}{P(F)} = \frac{100/800}{400/800} = \frac{100}{400} = \frac{1}{4} = 0.25$$

Example 7

The official tutorial for the programming language Python encourages users to make Monty Python references in code documentation, since the language is named after the group.

A university Computer Science Club conducted a survey of 500 students to determine their proficiency in Python and Java. The results showed that: 280 students know Python; 200 students know Java; 120 students know both languages.
A student is randomly selected from the surveyed group.

a. What is the probability that the selected student knows at least one of the two programming languages?

b. What is the probability that the student knows neither Python nor Java?

c. Given that a student knows Python, what is the probability that they also know Java?

d. Given that a student does not know Java, what is the probability that they know Python?

e. What is the probability that a randomly selected student knows exactly one of the two languages?

Solution

Let P = knows Python
J = knows Java

$$n(P) = 280, \quad n(J) = 200, \quad n(P \cap J) = 120, \quad n(S) = 500$$

a. What is the probability that the selected student knows at least one of the two programming languages?

$$\begin{aligned} P(P \cup J) &= P(P) + P(J) - P(P \cap J) \\ &= \frac{280}{500} + \frac{200}{500} - \frac{120}{500} \\ &= \frac{360}{500} \\ &= \frac{18}{25} \end{aligned}$$

b. What is the probability that the student knows neither Python nor Java?

$$P(\text{neither}) = 1 - P(P \cup J) = 1 - \frac{18}{25} = \frac{7}{25}$$

c. Given that a student knows Python, what is the probability that they also know Java?

$$P(J \mid P) = \frac{P(P \cap J)}{P(P)} = \frac{120/500}{280/500} = \frac{120}{280} = \frac{3}{7}$$

d. Given that a student does not know Java, what is the probability that they know Python?

$$\begin{aligned} P(P \mid J') &= \frac{P(P \cap J')}{P(J')} \\ &= \frac{280 - 120}{500 - 200} = \frac{160}{300} = \frac{8}{15} \end{aligned}$$

e. What is the probability that a randomly selected student knows exactly one of the two languages?

$$P(\text{only one}) = P(P \cup J) - P(P \cap J) = \frac{360}{500} - \frac{120}{500} = \frac{240}{500} = \frac{12}{25}$$

Example 8

In 2016, a California hacker registered his car with number plate NULL. In 2018, he discovered that he was now being sent all traffic tickets in the state for which the police officer had not entered the number plate, and was promptly billed $\$12,049$ for violations that he did not commit.

A data analysis team was assigned to investigate the issue, reviewing 10,000 traffic tickets issued in 2018. They found the following: 1,200 tickets were issued to vehicles with the license plate NULL; 2,500 tickets were issued where no license plate was recorded (marked as "NULL" in the database); 900 tickets appeared in both categories, meaning they were issued to vehicles with the license plate NULL and also marked as "NULL" in the system.

A randomly selected traffic ticket from the 10,000 reviewed is analyzed.

a. What is the probability that the ticket was either issued to the hacker's car (plate NULL) or recorded with a missing plate (NULL entry in the system)?

b. What is the probability that the ticket was neither issued to the hacker's car nor recorded with a missing plate?

c. If a ticket was recorded with a missing plate (NULL in the system), what is the probability that it was actually issued to the hacker's car (plate NULL)?

d. If a ticket was not issued to the hacker's car, what is the probability that it was still marked as NULL in the system?

e. What is the probability that a randomly selected ticket belongs only to one of these two categories (i.e., it was either issued to the hacker's car but not marked as NULL in the system, or it was marked as NULL but not issued to the hacker's car)?

Solution

Let H = ticket issued to hacker's car (plate "NULL")
M = ticket marked as "NULL" in the system (missing plate)

$$n(H) = 1200, \quad n(M) = 2500, \quad n(H \cap M) = 900, \quad n(S) = 10{,}000$$

a. What is the probability that the ticket was either issued to the hacker's car or recorded with a missing plate?

$$\begin{aligned} P(H \cup M) &= P(H) + P(M) - P(H \cap M) \\ &= \frac{1200}{10{,}000} + \frac{2500}{10{,}000} - \frac{900}{10{,}000} \\ &= \frac{2800}{10{,}000} = \frac{7}{25} \end{aligned}$$

b. What is the probability that the ticket was neither issued to the hacker's car nor recorded with a missing plate?

$$P(\text{neither}) = 1 - P(H \cup M) = 1 - \frac{7}{25} = \frac{18}{25}$$

c. If a ticket was recorded with a missing plate, what is the probability that it was actually issued to the hacker's car?

$$P(H \mid M) = \frac{P(H \cap M)}{P(M)} = \frac{900/10{,}000}{2500/10{,}000} = \frac{900}{2500} = \frac{9}{25}$$

d. If a ticket was not issued to the hacker's car, what is the probability that it was still marked as NULL in the system?

$$P(M \mid \bar{H}) = \frac{P(M \cap \bar{H})}{P(\bar{H})} = \frac{2500 - 900}{10{,}000 - 1200} = \frac{1600}{8800} = \frac{2}{11}$$

e. What is the probability that a randomly selected ticket belongs only to one of the two categories?

$$\begin{aligned} P(\text{only one}) &= P(H \cup M) - P(H \cap M) \\ &= \frac{2800}{10{,}000} - \frac{900}{10{,}000} = \frac{1900}{10{,}000} = \frac{19}{100} \end{aligned}$$

Bayes' Rule, and the Rule of Total Probability

Example 9

A group of bats in a wildlife sanctuary is monitored for a rare virus. Scientists know that: 80% of the bats belong to Species A, while the remaining 20% belong to Species B. In Species A, 5% of bats carry the virus. In Species B, 12% of bats carry the virus. A randomly selected bat is tested.

a. What is the probability that the bat carries the virus?

b. What is the probability that a bat is from Species A and does not carry the virus?

c. If a bat is found to have the virus, what is the probability that it belongs to Species B?

d. If a bat is virus-free, what is the probability that it belongs to Species A?

Solution

Let A = bat is from Species A, B = bat is from Species B
Let V = bat carries the virus, V' = bat does not carry the virus

$$\begin{aligned} P(A) &= 0.80, \quad & P(B) &= 0.20 \\ P(V \mid A) &= 0.05, \quad & P(V \mid B) &= 0.12 \\ P(V' \mid A) &= 0.95, \quad & P(V' \mid B) &= 0.88 \end{aligned}$$

a. What is the probability that the bat carries the virus?

$$\begin{aligned} P(V) &= P(V \mid A)P(A) + P(V \mid B)P(B) \\ &= (0.05)(0.80) + (0.12)(0.20) \\ &= 0.04 + 0.024 \\ &= 0.064 \end{aligned}$$

b. What is the probability that a bat is from Species A and does not carry the virus?

$$\begin{aligned} P(A \cap V') &= P(V' \mid A)P(A) \\ &= (0.95)(0.80) \\ &= 0.76 \end{aligned}$$

c. If a bat is found to have the virus, what is the probability that it belongs to Species B?

$$P(B \mid V) = \frac{P(B \cap V)}{P(V)} = \frac{P(V \mid B)P(B)}{P(V)} = \frac{(0.12)(0.20)}{0.064} = \frac{0.024}{0.064} = \frac{3}{8}$$

d. If a bat is virus-free, what is the probability that it belongs to Species A?

$$P(A \mid V') = \frac{P(A \cap V')}{P(V')} = \frac{P(V' \mid A)P(A)}{1 - P(V)} = \frac{(0.95)(0.80)}{0.936} = \frac{0.76}{0.936} = \frac{95}{117}$$

Example 10

A space agency has two types of temperature sensors installed on a space probe: Type X sensors make up 70% of the total, and Type Y sensors make up 30%. Type X sensors have a 3% chance of failing. Type Y sensors have a 7% chance of failing. A randomly selected sensor is checked.

a. What is the probability that a randomly selected sensor is faulty?

b. What is the probability that a randomly selected sensor is Type X and functions correctly?

c. If a sensor is found to be faulty, what is the probability that it is of Type Y?

d. If a sensor is working correctly, what is the probability that it is of Type X?

Solution

Let X = sensor is Type X, Y = sensor is Type Y
Let F = sensor is faulty, F' = sensor is not faulty

$$\begin{aligned} P(X) &= 0.70, \quad P(Y) = 0.30 \\ P(F \mid X) &= 0.03, \quad P(F \mid Y) = 0.07 \\ P(F' \mid X) &= 0.97, \quad P(F' \mid Y) = 0.93 \end{aligned}$$

a. What is the probability that a randomly selected sensor is faulty?

$$\begin{aligned} P(F) &= P(F \mid X)P(X) + P(F \mid Y)P(Y) \\ &= (0.03)(0.70) + (0.07)(0.30) \\ &= 0.021 + 0.021 \\ &= 0.042 \end{aligned}$$

b. What is the probability that a randomly selected sensor is Type X and functions correctly?

$$\begin{aligned} P(X \cap F') &= P(F' \mid X)P(X) \\ &= (0.97)(0.70) \\ &= 0.679 \end{aligned}$$

c. If a sensor is found to be faulty, what is the probability that it is of Type Y?

$$P(Y \mid F) = \frac{P(Y \cap F)}{P(F)} = \frac{P(F \mid Y)P(Y)}{P(F)} = \frac{(0.07)(0.30)}{0.042} = \frac{0.021}{0.042} = \frac{1}{2}$$

d. If a sensor is working correctly, what is the probability that it is of Type X?

$$P(X \mid F') = \frac{P(X \cap F')}{P(F')} = \frac{(0.97)(0.70)}{1 - 0.042} = \frac{0.679}{0.958} = \frac{679}{958}$$

Example 11

A school orders electronic calculators from two different suppliers: Supplier A provides 60% of the calculators, and Supplier B provides 40%. Supplier A's calculators have a 2% defect rate. Supplier B's calculators have a 5% defect rate. A randomly selected calculator is inspected.

a. What is the probability that a randomly selected calculator is defective?

b. What is the probability that a randomly selected calculator was supplied by Supplier A and is functioning properly?

c. If a calculator is found to be defective, what is the probability that it came from Supplier B?

d. If a calculator is not defective, what is the probability that it came from Supplier A?

Solution

Let A = calculator is from Supplier A, B = calculator is from Supplier B
Let D = calculator is defective, D' = calculator is not defective

$$\begin{aligned} P(A) &= 0.60, \quad & P(B) &= 0.40 \\ P(D \mid A) &= 0.02, \quad & P(D \mid B) &= 0.05 \\ P(D' \mid A) &= 0.98, \quad & P(D' \mid B) &= 0.95 \end{aligned}$$

a. What is the probability that a randomly selected calculator is defective?

$$\begin{aligned} P(D) &= P(D \mid A)P(A) + P(D \mid B)P(B) \\ &= (0.02)(0.60) + (0.05)(0.40) \\ &= 0.012 + 0.020 \\ &= 0.032 \end{aligned}$$

b. What is the probability that a randomly selected calculator was supplied by Supplier A and is functioning properly?

$$\begin{aligned} P(A \cap D') &= P(D' \mid A)P(A) \\ &= (0.98)(0.60) \\ &= 0.588 \end{aligned}$$

c. If a calculator is found to be defective, what is the probability that it came from Supplier B?

$$P(B \mid D) = \frac{P(D \mid B)P(B)}{P(D)} = \frac{(0.05)(0.40)}{0.032} = \frac{0.020}{0.032} = \frac{5}{8}$$

d. If a calculator is not defective, what is the probability that it came from Supplier A?

$$P(A \mid D') = \frac{P(A \cap D')}{P(D')} = \frac{(0.98)(0.60)}{1 - 0.032} = \frac{0.588}{0.968} = \frac{147}{242}$$

Independence and the Multiplication Rule

Example 12

At a fast food restaurant, customers at the drive-thru must choose one of three main meal options: Burger, Chicken, or Vegetarian. As a restaurant manager, you observe that: 50% of customers order a Burger; 30% order Chicken; and 20% order a Vegetarian meal. Assuming that each customer makes their choice independently, what is the probability that for the next three customers:

a. All order a Burger?

b. Two order Chicken, and one orders Vegetarian?

c. Each customer orders a different meal?

d. Exactly two order a Vegetarian meal?

e. All ordered Chicken, given that all ordered the same type of meal?

Solution

Let B = customer orders a Burger, C = customer orders Chicken, V = customer orders a Vegetarian meal

$$P(B) = 0.5, \quad P(C) = 0.3, \quad P(V) = 0.2$$

a. What is the probability that all three customers order a Burger?

$$P(BBB) = (0.5)^3 = 0.125$$

b. What is the probability that two order Chicken, and one orders Vegetarian?

There are 3 elements, but 2 of them repeat: CCV, CVC, VCC

$$P(\text{2 Chicken, 1 Vegetarian}) = \frac{3!}{2!} \cdot (0.3)^2 \cdot (0.2) = 0.054$$

c. What is the probability that each customer orders a different meal?

There are 3! possible arrangements: BCV, BVC, CBV, CVB, VBC, VCB

$$P(\text{all different}) = 3! \cdot (0.5)(0.3)(0.2) = 0.18$$

d. What is the probability that exactly two order a Vegetarian meal?

There are 3 such arrangements: VVB, VBV, BVV

$$P(\text{2 Vegetarian, 1 other}) = C_2^3 \cdot (0.2)^2 \cdot (1 - 0.2) = 0.096$$

e. What is the probability that all ordered Chicken, given that all ordered the same type of meal?

$$\begin{aligned} P(\text{All Chicken} \mid \text{All same}) &= \frac{P(CCC)}{P(BBB) + P(CCC) + P(VVV)} \\ &= \frac{(0.3)^3}{(0.5)^3 + (0.3)^3 + (0.2)^2} \\ &= \frac{0.027}{0.16} \\ &= \frac{27}{160} \\ &= 0.16875 \end{aligned}$$

Example 13

A fair coin is flipped, and a six-sided die is rolled.

a. What is the probability of getting heads on the coin and rolling a 3 on the die?

b. What is the probability of getting either heads or a 3 on the die?

c. What is the probability of not getting heads and not rolling a 3?

Solution

Let H = coin shows heads, T = coin shows tails
Let D_3 = die shows a 3, D' = die shows any number other than 3

$$P(H) = \frac{1}{2}, \quad P(D_3) = \frac{1}{6}, \quad P(D') = \frac{5}{6}$$

a. What is the probability of getting heads on the coin and rolling a 3 on the die?

$$P(H \cap D_3) = P(H) \cdot P(D_3) = \frac{1}{2} \cdot \frac{1}{6} = \frac{1}{12}$$

b. What is the probability of getting either heads or a 3 on the die?

$$\begin{aligned} P(H \cup D_3) &= P(H) + P(D_3) - P(H \cap D_3) \\ &= \frac{1}{2} + \frac{1}{6} - \frac{1}{12} \\ &= \frac{7}{12} \end{aligned}$$

c. What is the probability of not getting heads and not rolling a 3?

$$P(H' \cap D_3') = P(T \cap D') = P(T) \cdot P(D') = \frac{1}{2} \cdot \frac{5}{6} = \frac{5}{12}$$

Example 14

A math club has 10 students, 4 of whom are left-handed. A student is randomly chosen and then another is randomly chosen with replacement.

a. What is the probability that both chosen students are left-handed?

b. What is the probability that at least one of the selected students is left-handed?

c. If a student is found to be left-handed, what is the probability that the next student chosen is also left-handed?

Solution

Let L = student is left-handed, R = student is right-handed

$$P(L) = \frac{4}{10} = 0.4, \quad P(R) = \frac{6}{10} = 0.6$$

a. What is the probability that both chosen students are left-handed?

$$P(L \cap L) = P(L) \cdot P(L) = (0.4)^2 = 0.16$$

b. What is the probability that at least one of the selected students is left-handed?

$$P(\text{at least one } L) = 1 - P(R \cap R) = 1 - (0.6)^2 = 1 - 0.36 = 0.64$$

c. If a student is found to be left-handed, what is the probability that the next student chosen is also left-handed?

$$P(L \mid L) = P(L) = 0.4 \quad \text{selection is with replacement}$$

Example 15

A factory produces electronic chips, and each chip has an independent 2% chance of being defective.

a. If a customer buys three chips, what is the probability that all three are not defective?

b. What is the probability that at least one chip is defective?

c. If at least one chip is found to be defective, what is the probability that exactly one chip is defective?

Solution

Let D = chip is defective, D' = chip is not defective

$$P(D) = 0.02, \quad P(D') = 0.98$$

a. If a customer buys three chips, what is the probability that all three are not defective?

$$P(D' \cap D' \cap D') = (0.98)^3 = 0.9412$$

b. What is the probability that at least one chip is defective?

$$P(\text{at least one } D) = 1 - P(\text{no defective}) = 1 - (0.98)^3 = 0.0588$$

c. If at least one chip is found to be defective, what is the probability that exactly one chip is defective?

$$\begin{aligned} P(\text{exactly one } D \mid \text{at least one } D) &= \frac{P[(\text{exactly one } D) \cap (\text{at least one } D)]}{P(\text{at least one } D)} \\ &= \frac{P(\text{exactly one } D)}{P(\text{at least one } D)} \\ &= \frac{C_1^3 (0.02)^1 (0.98)^2}{0.0588} \\ &= \frac{0.0576}{0.0588} \\ &= 0.980 \end{aligned}$$

Example 16

A shelf has 3 mystery novels and 7 science fiction novels. A person randomly selects one book, reads it, and then picks another.

a. What is the probability that they first pick a mystery novel and then a science fiction novel?

b. What is the probability that at least one of the two books selected is a mystery novel?

c. Given that the first book selected was a mystery novel, what is the probability that the second book is also a mystery novel?

Solution

Let M = selects a mystery novel, S = selects a science fiction novel

$$\text{Total books} = 10, \quad P(M_1) = \frac{3}{10}, \quad P(S_1) = \frac{7}{10}$$

a. What is the probability that they first pick a mystery novel and then a science fiction novel?

$$P(M_1 \cap S_2) = P(M_1) \cdot P(S_2 \mid M_1) = \frac{3}{10} \cdot \frac{7}{9} = \frac{21}{90} = \frac{7}{30}$$

b. What is the probability that at least one of the two books selected is a mystery novel?

We use the complement: both books are science fiction.

$$P(\text{at least one } M) = 1 - P(S_1 \cap S_2) = 1 - \frac{7}{10} \cdot \frac{6}{9} = 1 - \frac{42}{90} = \frac{48}{90} = \frac{8}{15}$$

c. Given that the first book selected was a mystery novel, what is the probability that the second book is also a mystery novel?

$$P(M_2 \mid M_1) = \frac{2}{9}$$

Example 17

A deck contains 5 red cards and 5 black cards. Two cards are drawn without replacement.

a. What is the probability that both drawn cards are red?

b. What is the probability that at least one of the two drawn cards is red?

c. If the first card drawn is red, what is the probability that the second card is also red?

Solution

Let R = red card, B = black card

$$\text{Total cards} = 10, \quad \text{Red cards} = 5, \quad \text{Black cards} = 5$$

a. What is the probability that both drawn cards are red?

$$P(R_1 \cap R_2) = P(R_1) \cdot P(R_2 \mid R_1) = \frac{5}{10} \cdot \frac{4}{9} = \frac{20}{90} = \frac{2}{9}$$

b. What is the probability that at least one of the two drawn cards is red?
Use the complement: both cards are black.

$$P(\text{at least one } R) = 1 - P(B_1 \cap B_2) = 1 - \left(\frac{5}{10} \cdot \frac{4}{9}\right) = 1 - \frac{20}{90} = \frac{70}{90} = \frac{7}{9}$$

c. If the first card drawn is red, what is the probability that the second card is also red?

$$P(R_2 \mid R_1) = \frac{4}{9}$$

Example 18

A bag contains 8 blue marbles and 2 red marbles. You draw two marbles without replacement.

a. What is the probability of drawing a red marble first, then a blue marble?

b. What is the probability that at least one of the drawn marbles is red?

c. If the first marble drawn was red, what is the probability that the second marble is also red?

Solution

Let R = red marble, B = blue marble

$$\text{Total marbles} = 10, \quad \text{Red} = 2, \quad \text{Blue} = 8$$

a. What is the probability of drawing a red marble first, then a blue marble?

$$P(R_1 \cap B_2) = P(R_1) \cdot P(B_2 \mid R_1) = \frac{2}{10} \cdot \frac{8}{9} = \frac{16}{90} = \frac{8}{45}$$

b. What is the probability that at least one of the drawn marbles is red?

Use the complement: both marbles are blue.

$$P(\text{at least one } R) = 1 - P(B_1 \cap B_2) = 1 - \left(\frac{8}{10} \cdot \frac{7}{9}\right) = 1 - \frac{56}{90} = \frac{34}{90} = \frac{17}{45}$$

c. If the first marble drawn was red, what is the probability that the second marble is also red?

$$P(R_2 \mid R_1) = \frac{1}{9}$$