# L7. Probability Mass Functions

**Example 1**

Let $X$ be a random variable. Verify that the following is a probability mass function and calculate the requested probabilities.

| $x$ | $-2$ | $-1$ | $0$ | $1$ | $2$ |
|---|---|---|---|---|---|
| $f(x)$ | 0.2 | 0.4 | 0.1 | 0.2 | 0.1 |

a. $P(X \leq 2)$

b. $P(X > -2)$

c. $P(-1 \leq X \leq 1)$

d. $P(X \leq -1 \text{ or } X = 2)$

**Solution**

**Verification:** A function $P(X = x)$ is a valid probability mass function (pmf) if:

1. $P(X = x) \geq 0$ for all $x$.
2. $\sum_x P(X = x) = 1$.

Check on Non-negativity: $f(x) = P(X = x_i) \geq 0 \forall x_i$

Check sum: $\sum f(x) = \sum P(X = x_i) = 0.2 + 0.4 + 0.1 + 0.2 + 0.1 = 1$

Both conditions are satisfied $\therefore f(x)$ is a valid pmf.

a. $P(X \leq 2)$

$$\begin{aligned} P(X \leq 2) &= P(X = -2) + P(X = 1) + P(X = 0) + P(X = 1) + P(X = 2) \\ &= 0.2 + 0.4 + 0.1 + 0.2 + 0.1 \\ &= 1 \end{aligned}$$

b. $P(X > -2)$

$$\begin{aligned} P(X > -2) &= P(X = 1) + P(X = 0) + P(X = 1) + P(X = 2) \\ &= 0.4 + 0.1 + 0.2 + 0.1 \\ &= 0.8 \end{aligned}$$

c. $P(-1 \le X \le 1)$

$$\begin{aligned} P(-1 \le X \le 1) &= P(X = 1) + P(X = 0) + P(X = 1) \\ &= 0.4 + 0.1 + 0.2 \\ &= 0.7 \end{aligned}$$

d. $P(X \le -1 \text{ or } X = 2)$

$$\begin{aligned} P(X \le -1 \text{ or } X = 2) &= P(X \le 1) + P(X = 2) \\ &= (0.2 + 0.4) + 0.1 \\ &= 0.7 \end{aligned}$$

**Example 2**

Let $X$ be a random variable. Verify that the following is a probability mass function and calculate the requested probabilities.

$$f(x) = \frac{2x+1}{25} \qquad x = 0, 1, 2, 3, 4$$

a. $P(X = 4)$

b. $P(X \le 1)$

c. $P(2 \le X < 4)$

d. $P(X > -10)$

**Solution**

**Verification:** A function $P(X = x)$ is a valid probability mass function (pmf)

Check on Non-negativity:

$$f(x) = \frac{2x+1}{25} \ge 0 \quad \forall x \in \{0, 1, 2, 3, 4\}$$

Check sum:

$$\begin{aligned} \sum f(x) &= \frac{2(0)+1}{25} + \frac{2(1)+1}{25} + \frac{2(2)+1}{25} + \frac{2(3)+1}{25} \\ &= \frac{1+3+5+7+9}{25} \\ &= 1 \end{aligned}$$

Both conditions are satisfied $\therefore f(x)$ is a valid pmf.

a. $P(X = 4)$

$$P(X = 4) = \frac{2(4) + 1}{25} = \frac{9}{25}$$

b. $P(X \leq 1)$

$$P(X \leq 1) = P(X = 0) + P(X = 1) = \frac{1}{25} + \frac{3}{25} = \frac{4}{25}$$

c. $P(2 \leq X < 4)$

$$P(2 \leq X < 4) = P(X = 2) + P(X = 3) = \frac{5}{25} + \frac{7}{25} = \frac{12}{25}$$

d. $P(X > -10)$

$$\begin{aligned} P(X > -10) &= P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4) \\ &= \frac{1 + 3 + 5 + 7 + 9}{25} \\ &= 1 \end{aligned}$$

### Example 3

In a semiconductor manufacturing process, three wafers from a lot are tested. Each wafer is classified as pass or fail. Assume that the probability that a wafer passes the test is 0.8 and that wafers are independent. Determine the probability mass function of the number of wafers from a lot that pass the test.

### Solution

Let $X$ = the number of wafers in our sample that pass the test; $p = 0.8$
Then the pmf is

| $X$ | $P(X = x_i)$ |
|---|---|
| 0 | $P(X = 0) = C_0^3(0.8)^0(0.2)^3 = 0.008$ |
| 1 | $P(X = 1) = C_1^3(0.8)^1(0.2)^2 = 0.096$ |
| 2 | $P(X = 2) = C_2^3(0.8)^2(0.2)^1 = 0.384$ |
| 3 | $P(X = 3) = C_3^3(0.8)^3(0.2)^0 = 0.512$ |

**Example 4**

An assembly consists of two mechanical components. Suppose that the probabilities that the first and second components meet specifications are 0.95 and 0.98, respectively. Assume that the components are independent. Determine the probability mass function of the number of components in the assembly that meet specifications.

**Solution**

Let $X$ = the number of components in the assembly that meet specifications. The probability that the first component meets specifications is 0.95, and for the second it is 0.98. Assume the components are independent.

Then the pmf is:

| $X$ | $P(X = x_i)$ |
|---|---|
| 0 | $P(X = 0) = (1 - 0.95)(1 - 0.98) = 0.001$ |
| 1 | $P(X = 1) = (0.95)(1 - 0.98) + (1 - 0.95)(0.98) = 0.068$ |
| 2 | $P(X = 2) = (0.95)(0.98) = 0.931$ |

**Example 5**

An urn contains 11 chips; 3 are white, 3 are red, and 5 are black. Take 3 chips out of the urn at random, and without replacement. You win \$1 for each red chip that you get and lose a \$1 for each white that you get in your selection. Let $X$ represent the amount of money that you win. Determine the mass function of $X$.

**Solution**

Let $X$ be the amount of money you win in the experiment.

Possible values of $X$: $-3, -2, -1, 0, 1, 2, 3$ and $n(S) = C_3^{11} = 165$

$X = -3$: (3 white chips)

$$P(X = -3) = \frac{C_3^3}{C_3^{11}} = \frac{1}{165}$$

$X = -2$: (2 white, 1 black)

$$P(X = -2) = \frac{C_2^3 \cdot C_1^5}{C_3^{11}} = \frac{15}{165}$$

$X = -1$: (2 white, 1 red) or (1 white, 2 black)

$$P(X = -1) = \frac{C_2^3 \cdot C_1^3 + C_1^3 \cdot C_2^5}{C_3^{11}} = \frac{9 + 30}{165} = \frac{39}{165}$$

$X = 0$: (1 red, 1 white, 1 black) or (3 black)

$$P(X = 0) = \frac{C_1^3 \cdot C_1^3 \cdot C_1^5 + C_3^5}{C_3^{11}} = \frac{45 + 10}{165} = \frac{45}{165}$$

$X = 1$: (2 red, 1 white) or (1 red, 2 black)

$$P(X = 1) = \frac{C_2^3 \cdot C_1^3 + C_1^3 \cdot C_2^5}{C_3^{11}} = \frac{9 + 30}{165} = \frac{39}{165}$$

$X = 2$: (2 red, 1 black)

$$P(X = 2) = \frac{C_2^3 \cdot C_1^5}{C_3^{11}} \frac{15}{165}$$

**Case** $X = 3$: (3 red)

$$P(X = 3) = \frac{C_3^3}{C_3^{11}} \frac{1}{165}$$

The pmf is

| $X$ | $-3$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ |
|---|---|---|---|---|---|---|---|
| $P(X = x_i)$ | $\frac{1}{165}$ | $\frac{15}{165}$ | $\frac{39}{165}$ | $\frac{55}{165}$ | $\frac{39}{165}$ | $\frac{15}{165}$ | $\frac{1}{165}$ |

### Example 6

Roll a red die and a green die. Let the random variable, $X$, be the larger of the two numbers if they are different and the common value if they are the same. Generate an expression for the probability mass function for $X$.

### Solution

Let $X$ be the maximum of the two rolls.

Possible values of $X$: $1, 2, 3, 4, 5, 6$ and $n(S) = 6 \times 6 = 36$

$X = 1$: only when both dice show 1

$$P(X = 1) = \frac{1}{36}$$

$X = 2$: occurs for outcomes (1,2), (2,1), (2,2)

$$P(X = 2) = \frac{3}{36}$$

$X = 3$: occurs for outcomes (1,3), (3,1), (2,3), (3,2), (3,3)

$$P(X = 3) = \frac{5}{36}$$

A similar approach is used to compute $X = 4, X = 5$, and $X = 6$. The full pmf is

| $X$ | $1$ | $2$ | $3$ | $4$ | $5$ | $6$ |
|---|---|---|---|---|---|---|
| $P(X = x)$ | $\frac{1}{36}$ | $\frac{3}{36}$ | $\frac{5}{36}$ | $\frac{7}{36}$ | $\frac{9}{36}$ | $\frac{11}{36}$ |