L8. Expected Value and Variance of Discrete Random Variables

Example 1

Let X be a random variable with the following probability mass function.

x	-2	-1	0	1	2
$f(x)$	0.2	0.4	0.1	0.2	0.1

a. Determine the expected value of X.

b. Determine the variance of X

c. What is the standard deviation of X?

Solution

Here is the original augmented with additional columns to facilitate our calculations.

x	$P(X=x)$	$x \cdot P(X=x)$	$x^2 \cdot P(X=x)$
-2	0.2	-0.4	0.8
-1	0.4	-0.4	0.4
0	0.1	0	0
1	0.2	0.2	0.2
2	0.1	0.2	0.4

a. Expected value:

$$\begin{aligned}\mathbb{E}[X] &= \sum x_i \cdot P(X = x_i) \\ &= (-2)(0.2) + (-1)(0.4) + (0)(0.1) + (1)(0.2) + (2)(0.1) \\ &= -0.4 - 0.4 + 0 + 0.2 + 0.2 \\ &= -0.4\end{aligned}$$

b. Variance:

$$\begin{aligned}\mathbb{V}[X] &= \mathbb{E}[X^2] - (\mathbb{E}[X])^2 \\ &= \sum x_i^2 \cdot P(X = x_i) - \left(\sum x_i \cdot P(X = x_i)\right)^2 \\ &= [(-2)^2(0.2) + (-1)^2(0.4) + 0 + (1)^2(0.2) + 2^2(0.1)] - (-0.4)^2 \\ &= 1.64\end{aligned}$$

c. Standard deviation:

$$\mathbb{S}[X] = \sqrt{\mathbb{V}[X]} = \sqrt{1.64} \approx 1.28$$

Example 2

In a semiconductor manufacturing process, three wafers from a lot are tested. Each wafer is classified as pass or fail. Assume that the probability that a wafer passes the test is 0.8 and that wafers are independent.

a. Determine the probability mass function of the number of wafers from a lot that pass the test.

b. Determine $E[X]$ and interpret it in the context of the problem.

c. Determine $V[X]$ and $S[X]$.

Solution

Here is the original augmented with additional columns to facilitate our calculations.

x	$P(X=x)$	$x \cdot P(X=x)$	$x^2 \cdot P(X=x)$
0	0.008	0	0
1	0.096	0.096	0.096
2	0.384	0.768	1.536
3	0.512	1.536	4.608

a. Probability mass function:

$$\begin{aligned} P(X=0) &= C_0^3(0.8)^0(0.2)^3 = 0.008 \\ P(X=1) &= C_1^3(0.8)^1(0.2)^2 = 0.096 \\ P(X=2) &= C_2^3(0.8)^2(0.2)^1 = 0.384 \\ P(X=3) &= C_3^3(0.8)^3(0.2)^0 = 0.512 \end{aligned}$$

b. Expected value:

$$\begin{aligned} \mathbb{E}[X] &= \sum x_i \cdot P(X=x_i) \\ &= 0 \cdot 0.008 + 1 \cdot 0.096 + 2 \cdot 0.384 + 3 \cdot 0.512 \\ &= 0 + 0.096 + 0.768 + 1.536 \\ &= 2.4 \end{aligned}$$

Interpretation: On average, we expect 2.4 out of 3 wafers in a lot to pass the test in the long run, under repeated sampling in this manufacturing process.

c. Variance:

$$\begin{aligned} \mathbb{V}[X] &= \sum x_i^2 \cdot P(X=x_i) - (\mathbb{E}[X])^2 \\ &= 0 + 0.096 + 1.536 + 4.608 - (2.4)^2 \\ &= 6.24 - 5.76 \\ &= 0.48 \end{aligned}$$

Standard deviation:

$$\mathbb{S}[X] = \sqrt{\mathbb{V}[X]} = \sqrt{0.48} \approx 0.6928$$

Example 3

An assembly consists of two mechanical components. Suppose that the probabilities that the first and second components meet specifications are 0.95 and 0.98, respectively. Assume that the components are independent.

a. Determine the probability mass function of the number of components in the assembly that meet specifications.

b. Determine $E[X]$ and interpret it in the context of the problem.

c. Determine $V[X]$ and $S[X]$.

Solution

Here is the original augmented with additional columns to facilitate our calculations.

x	$P(X=x)$	$x \cdot P(X=x)$	$x^2 \cdot P(X=x)$
0	0.001	0	0
1	0.068	0.068	0.068
2	0.931	1.862	3.724

a. Probability mass function:

$$\begin{aligned} P(X=0) &= (1-0.95)(1-0.98) = 0.001 \\ P(X=1) &= (0.95)(1-0.98) + (1-0.95)(0.98) = 0.068 \\ P(X=2) &= (0.95)(0.98) = 0.931 \end{aligned}$$

b. Expected value:

$$\begin{aligned} \mathbb{E}[X] &= \sum x_i \cdot P(X=x_i) \\ &= 0 \cdot 0.001 + 1 \cdot 0.068 + 2 \cdot 0.931 \\ &= 0 + 0.068 + 1.862 = 1.93 \end{aligned}$$

Interpretation: On average, we expect 1.93 out of 2 component to meet specifications in the long run.

c. Variance:

$$\begin{aligned} \mathbb{V}[X] &= \sum x_i^2 \cdot P(X=x_i) - (\mathbb{E}[X])^2 \\ &= 0 + 0.068 + 3.724 - (1.93)^2 \\ &= 3.792 - 3.7249 = 0.0671 \end{aligned}$$

Standard deviation:

$$\mathbb{S}[X] = \sqrt{\mathbb{V}[X]} = \sqrt{0.0671} \approx 0.26$$

Example 4

An urn contains 11 chips; 3 are white, 3 are red, and 5 are black. Take 3 chips out of the urn at random, and without replacement. You win \$1 for each red chip that you get and lose a \$1 for each white that you get in your selection. Let X represent the amount of money that you win.

a. Determine the mass function of X.

b. Determine $E[X]$ and interpret it in the context of the problem.

c. Determine $V[X]$ and $S[X]$.

Solution

Here is the original augmented with additional columns to facilitate our calculations.

X	$P(X = x_i)$	$x \cdot P(X = x_i)$	$x^2 \cdot P(X = x_i)$
-3	$\frac{1}{165}$	$-\frac{3}{165}$	$\frac{9}{165}$
-2	$\frac{15}{165}$	$-\frac{30}{165}$	$\frac{60}{165}$
-1	$\frac{39}{165}$	$-\frac{39}{165}$	$\frac{39}{165}$
0	$\frac{55}{165}$	0	0
1	$\frac{39}{165}$	$\frac{39}{165}$	$\frac{39}{165}$
2	$\frac{15}{165}$	$\frac{30}{165}$	$\frac{60}{165}$
3	$\frac{1}{165}$	$\frac{3}{165}$	$\frac{9}{165}$

a. The probability mass function is given above.

b. Expected value:

$$\mathbb{E}[X] = \sum x_i \cdot P(X = x_i)$$
$$= \frac{-3 - 30 - 39 + 0 + 39 + 30 + 3}{165} = \frac{0}{165} = 0$$

Interpretation: On average, you neither win nor lose money in this game. The expected value is \$0.

c. Variance:

$$\mathbb{E}[X^2] = \sum x_i^2 \cdot P(X = x_i) = \frac{9 + 60 + 39 + 0 + 39 + 60 + 9}{165} = \frac{216}{165}$$
$$\mathbb{V}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \frac{216}{165} - 0 = \frac{216}{165} \approx 1.309$$

Standard deviation:

$$\mathbb{S}[X] = \sqrt{\mathbb{V}[X]} = \sqrt{\frac{216}{165}} \approx 1.144$$