

Class Exercise 5 — Solutions

1. Search and Rescue Operations

A group of 56 volunteers has assembled to assist in a search and rescue operation in a national park. The incident coordinator needs to divide the volunteers into four field search parties: Team Alpha (15 members), Team Bravo (12 members), Team Charlie (11 members), and Team Delta (10 members). The remaining volunteers will stay behind at Headquarters (HQ) to coordinate communications and logistics.

- a. In how many ways can the 56 volunteers be assigned to the four search parties and the headquarters team? (You may leave your answer in factorial or combination notation.)

Solution

The number of volunteers remaining at Headquarters is $56 - (15 + 12 + 11 + 10) = 56 - 48 = 8$.

Dividing $n = 56$ distinct individuals into 5 distinct groups of sizes 15, 12, 11, 10, 8 gives:

$$\binom{56}{15, 12, 11, 10, 8} = \frac{56!}{15! 12! 11! 10! 8!} \quad \left(\text{or } \binom{56}{15} \binom{41}{12} \binom{29}{11} \binom{18}{10} \binom{8}{8} \right).$$

- b. Two of the volunteers, Maria and David, are experienced radio dispatchers and must both be assigned to the headquarters team. In how many ways can the assignments now be made?

Solution

Since Maria and David are pre-assigned to HQ, there are $8 - 2 = 6$ spots remaining to be filled at HQ, and $56 - 2 = 54$ remaining volunteers to be distributed:

$$\binom{54}{15, 12, 11, 10, 6} = \frac{54!}{15! 12! 11! 10! 6!} \quad \left(\text{or } \binom{54}{15} \binom{39}{12} \binom{27}{11} \binom{16}{10} \binom{6}{6} \right).$$

- c. If assignments are made completely at random, what is the probability that both Maria and David end up assigned to the headquarters team?

Solution

Method 1 (Ratio of favorable to total assignments):

$$P(\text{both at HQ}) = \frac{\frac{54!}{15! 12! 11! 10! 6!}}{\frac{56!}{15! 12! 11! 10! 8!}} = \frac{54!}{56!} \cdot \frac{8!}{6!} = \frac{8 \cdot 7}{56 \cdot 55} = \frac{1}{55} \approx 0.01818 \quad (1.82\%).$$

Method 2 (Choosing 2 spots out of 8 at HQ): There are 8 available spots at HQ out of 56 total spots:

$$P(\text{both at HQ}) = \frac{\binom{8}{2}}{\binom{56}{2}} = \frac{\frac{8 \cdot 7}{2}}{\frac{56 \cdot 55}{2}} = \frac{28}{1540} = \frac{1}{55} \approx 0.01818.$$

(Equivalently, sequentially: $\frac{8}{56} \cdot \frac{7}{55} = \frac{1}{7} \cdot \frac{7}{55} = \frac{1}{55}$.)

2. The Birthday Problem

Assume there are 365 days in a year and all birthdays are equally likely.

- a. In a group of 60 randomly selected people, what is the probability that at least two people share the same birthday?

Solution (2 points)

This is the complement of the event that “all 60 people have unique birthdays”.

$$P(\text{at least 2 share}) = 1 - P(\text{all unique})$$

The total number of ways to assign birthdays to 60 people is 365^{60} .

The number of ways to assign unique birthdays to 60 people is the permutation:

$$P(\text{all unique}) = \frac{{}_{365}P_{60}}{365^{60}} = \frac{365 \cdot 364 \cdot \dots \cdot 306}{365^{60}} = \frac{365!}{305! \cdot 365^{60}} \approx 0.005877$$

$$\begin{aligned} P(\text{at least 2 share}) &= 1 - \frac{{}_{365}P_{60}}{365^{60}} \\ &= 1 - 0.005877 \\ &= \mathbf{0.9941} \end{aligned}$$

- b. You are in a group of 60 randomly selected people. What is the probability that no one else in the group shares your birthday?

Solution (2 points)

Let YB = your birthday.

$$P(YB) = \frac{1}{365} \quad \Rightarrow \quad P(YB') = \frac{364}{365}$$

No one else in the group shares your birthday means that we need all 59 other people to not have your birthday.

$$P(\text{all 59 miss your birthday}) = \left(\frac{364}{365}\right)^{59} \approx \mathbf{0.8505}$$