

Class Exercise 6 – Solutions

1. Agentic swarm

Masterbot manages 22 AI agents residing in my laptop. There are four distinct group projects, which need 9, 5, 4 and 4 agents respectively. Every agent is assigned to exactly one project.

- a. In how many ways can the 22 agents be assigned to the four projects?

Solution. We fill the projects one at a time. We choose the 9 agents for the first project, then 5 of the remaining 13, then 4 of the remaining 8, and the last 4 agents form the final team:

$$\binom{22}{9} \binom{13}{5} \binom{8}{4} \binom{4}{4} = \frac{22!}{9! 5! 4! 4!} = 44\,812\,567\,800.$$

The two 4-agent projects are *different* projects, so we do **not** divide by $2!$.

- b. Three particular agents refuse to work with one another, so no two of these three may be on the same team. In how many ways can the assignment be done now?

Solution. Call the three agents X, Y, Z . They must go to three *different* projects, so exactly one project receives none of them. We place X, Y, Z first, in $3!$ ways once we know which three projects they occupy. The remaining 19 agents then fill the remaining spots.

- The 9-team gets none of them. The remaining spots are 9, 4, 3, 3:

$$3! \cdot \frac{19!}{9! 4! 3! 3!} = 6 \cdot 387\,987\,600.$$

- The 5-team gets none of them. The remaining spots are 8, 5, 3, 3:

$$3! \cdot \frac{19!}{8! 5! 3! 3!} = 6 \cdot 698\,377\,680.$$

- One of the two 4-teams gets none of them (2 choices). The remaining spots are 8, 4, 3, 4:

$$2 \cdot 3! \cdot \frac{19!}{8! 4! 3! 4!} = 2 \cdot 6 \cdot 872\,972\,100.$$

Adding the three cases:

$$6(387\,987\,600 + 698\,377\,680 + 1\,745\,944\,200) = 16\,993\,856\,880.$$

Check: $16\,993\,856\,880 / 44\,812\,567\,800 \approx 0.379$. This agrees with the direct probability that X, Y, Z land on different teams:

$$\sum_{\text{distinct } i, j, k} \frac{n_i}{22} \cdot \frac{n_j}{21} \cdot \frac{n_k}{20} = \frac{6(9 \cdot 5 \cdot 4 + 9 \cdot 5 \cdot 4 + 9 \cdot 4 \cdot 4 + 5 \cdot 4 \cdot 4)}{22 \cdot 21 \cdot 20} = \frac{3504}{9240} \approx 0.379.$$

2. **Poker hand**

A poker hand consists of 5 cards dealt from a standard 52-card deck. A hand is **Two Pairs** when it contains two cards of one rank, two cards of a second, different rank, and a fifth card whose rank differs from both. For example: $5\heartsuit, 5\spadesuit, Q\clubsuit, Q\spadesuit, 10\diamondsuit$. What is the probability that a randomly dealt poker hand is Two Pairs?

Solution. There are $\binom{52}{5} = 2\,598\,960$ equally likely hands. We count the Two Pairs hands in steps.

- Choose the two ranks for the pairs: $\binom{13}{2} = 78$. The order does not matter, since $\{5, Q\}$ is the same as $\{Q, 5\}$.
- Choose two suits for each pair: $\binom{4}{2}\binom{4}{2} = 36$.
- Choose the fifth card from the other 11 ranks: $11 \cdot 4 = 44$.

$$P(\text{Two Pairs}) = \frac{\binom{13}{2}\binom{4}{2}^2 \cdot 44}{\binom{52}{5}} = \frac{123\,552}{2\,598\,960} = \frac{198}{4165} \approx 0.0475.$$

3. **Koala thieves**

In 2006, thieves planning to steal a koala from a zoo in Australia had to change their minds when it proved too vicious to abduct. The perpetrators later confessed that their original plan was to swap the koala for drugs, but after it scratched the hell out of them, they stole a 1.4-metre crocodile instead, which they claimed was a lot easier.

- A witness saw a car speeding away from the zoo. Its licence plate had 7 symbols: a letter (V or W), then the three digits 4, 7, 8 in some order, then the three letters A, C, E in some order. The witness could not remember the order of the digits or of the last three letters. How many plates would need to be checked to be sure of including the suspect car?

Solution. The first letter can be chosen in 2 ways. The digits 4, 7, 8 can be arranged in $3!$ ways, and the letters A, C, E in $3!$ ways:

$$2 \cdot 3! \cdot 3! = 72 \text{ plates.}$$

- In other parts of Australia, a licence plate has seven symbols in the pattern

digit, letter, letter, letter, digit, digit, digit,

where a letter is any of the 26 letters A–Z and a digit is any of 0–9. Assume all such plates are equally likely.

- What is the probability that all seven symbols are different?

Solution. There are $10^4 \cdot 26^3 = 175\,760\,000$ plates in total. For all symbols to be different, the four digits must be distinct and the three letters must be distinct:

$$P = \frac{(10 \cdot 9 \cdot 8 \cdot 7)(26 \cdot 25 \cdot 24)}{10^4 \cdot 26^3} = \frac{78\,624\,000}{175\,760\,000} = \frac{378}{845} \approx 0.4473.$$

- ii. What is the probability that all seven symbols are different *and* the first digit (the leftmost symbol) is the largest of the four digits?

Solution. We choose the set of 4 distinct digits in $\binom{10}{4}$ ways. The largest must go in the first position, and the other three can be arranged in $3!$ ways. The letters are chosen as before:

$$P = \frac{\binom{10}{4} \cdot 3! \cdot (26 \cdot 25 \cdot 24)}{10^4 \cdot 26^3} = \frac{19\,656\,000}{175\,760\,000} = \frac{189}{1690} \approx 0.1118.$$

(Equivalently, when the four digits are distinct, each of them is equally likely to be the largest, so the answer is $\frac{1}{4} \cdot \frac{378}{845}$.)

4. Cards

A full deck of 52 cards is shuffled so that all $52!$ orderings are equally likely.

- a. What is the probability that the four Aces are in four consecutive positions in the deck? (You do not have to convert the answer to a decimal.)

Solution. Glue the four Aces into a single block. That leaves $48 + 1 = 49$ objects to arrange, in $49!$ ways, and the Aces can be ordered inside the block in $4!$ ways:

$$P = \frac{4! \cdot 49!}{52!} = \frac{24}{52 \cdot 51 \cdot 50} = \frac{1}{5525} \approx 0.000181.$$

- b. What is the probability that the $A_{\clubsuit}$ is not immediately above the $K_{\clubsuit}$ **and** the $A_{\spadesuit}$ is not immediately above the $K_{\spadesuit}$?

Solution. Let E_1 be the event that $A_{\clubsuit}$ is immediately above $K_{\clubsuit}$, and E_2 the event that $A_{\spadesuit}$ is immediately above $K_{\spadesuit}$.

For E_1 , glue “ $A_{\clubsuit}$ on top of $K_{\clubsuit}$ ” into one block. This gives 51 objects, so

$$P(E_1) = \frac{51!}{52!} = \frac{1}{52}, \quad P(E_2) = \frac{1}{52}.$$

For both events, glue both blocks. This gives 50 objects, so

$$P(E_1 \cap E_2) = \frac{50!}{52!} = \frac{1}{52 \cdot 51} = \frac{1}{2652}.$$

By inclusion–exclusion,

$$P(\overline{E_1} \cap \overline{E_2}) = 1 - P(E_1) - P(E_2) + P(E_1 \cap E_2) = 1 - \frac{2}{52} + \frac{1}{2652} = \frac{2551}{2652} \approx 0.9619.$$