

Class Exercise

1. Independence

If A and B are independent events prove that A' and B' are also independent.

Solution:

By the definition of independent events, $P(A \cap B) = P(A)P(B)$. We need to show that $P(A' \cap B') = P(A')P(B')$. Using De Morgan's laws and the basic axioms of probability:

$$\begin{aligned}
 P(A' \cap B') &= P((A \cup B)') \\
 &= 1 - P(A \cup B) \\
 &= 1 - (P(A) + P(B) - P(A \cap B)) \\
 &= 1 - P(A) - P(B) + P(A)P(B) \\
 &= (1 - P(A)) - P(B)(1 - P(A)) \\
 &= (1 - P(A))(1 - P(B)) \\
 &= P(A')P(B')
 \end{aligned}$$

Thus, A' and B' are also independent.

2. A group of girls at a school are taking Advanced Cantonese which come in two modules: C1 and C2. Each girl takes only module C1, or only module C2, or both C1 and C2. The probability that a girl is taking C2 given that she is taking C1 is 0.2. The probability that a girl is taking C1 given that she is taking C2 is 0.33

Find the probability that a girl selected at random

Solution Setup:

Let $P(C_1)$ and $P(C_2)$ be the probabilities of taking module C1 and C2, respectively. Since every girl takes at least one module, $P(C_1 \cup C_2) = 1$. We are given two conditional probabilities: 1) $P(C_2|C_1) = \frac{P(C_1 \cap C_2)}{P(C_1)} = 0.2 \implies P(C_1) = 5P(C_1 \cap C_2)$ 2) $P(C_1|C_2) = \frac{P(C_1 \cap C_2)}{P(C_2)} = 0.33 \implies P(C_2) = \frac{1}{0.33}P(C_1 \cap C_2) = \frac{100}{33}P(C_1 \cap C_2)$

Using the addition rule for probabilities:

$$\begin{aligned}
 P(C_1 \cup C_2) &= P(C_1) + P(C_2) - P(C_1 \cap C_2) \\
 1 &= 5P(C_1 \cap C_2) + \frac{100}{33}P(C_1 \cap C_2) - P(C_1 \cap C_2) \\
 1 &= \left(4 + \frac{100}{33}\right)P(C_1 \cap C_2) \\
 1 &= \frac{232}{33}P(C_1 \cap C_2) \\
 P(C_1 \cap C_2) &= \frac{33}{232}
 \end{aligned}$$

- (a) is taking both C1 and C2.

Solution:

This is exactly the intersection $P(C_1 \cap C_2)$.

$$P(C_1 \cap C_2) = \frac{33}{232} \approx 0.1422$$

- (b) is taking only C1.

Solution:

Taking only C1 corresponds to $P(C_1 \cap C'_2)$, which is $P(C_1) - P(C_1 \cap C_2)$.

$$P(C_1 \cap C'_2) = 5 \left(\frac{33}{232} \right) - \frac{33}{232} = 4 \left(\frac{33}{232} \right) = \frac{132}{232} = \frac{33}{58} \approx 0.5690$$

3. Roll a fair die 10 times. Compute the probability that you get

- (a) at least one 6.

Solution:

It is easier to find the complement (rolling no 6s in 10 rolls).

$$P(\text{at least one 6}) = 1 - P(\text{no 6}) = 1 - \left(\frac{5}{6} \right)^{10} \approx 0.8385$$

- (b) at least one 6 and at least one 5.

Solution:

Let A be the event of rolling no 6s, and B be the event of rolling no 5s. We want $1 - P(A \cup B)$. Using the addition rule: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.
 $P(A) = \left(\frac{5}{6} \right)^{10}$ $P(B) = \left(\frac{5}{6} \right)^{10}$ $P(A \cap B)$ is the probability of rolling neither a 5 nor a 6:
 $\left(\frac{4}{6} \right)^{10}$

$$P(\text{at least one 6 and at least one 5}) = 1 - \left[2 \left(\frac{5}{6} \right)^{10} - \left(\frac{4}{6} \right)^{10} \right] \approx 0.6943$$

- (c) three 1's, two 2's, and five 3's.

Solution:

This follows a multinomial distribution:

$$P(3 \text{ 1s, } 2 \text{ 2s, } 5 \text{ 3s}) = \frac{10!}{3!2!5!} \left(\frac{1}{6} \right)^3 \left(\frac{1}{6} \right)^2 \left(\frac{1}{6} \right)^5 = 2520 \left(\frac{1}{6} \right)^{10} \approx 0.0000417$$

- (d) at least one number occurring exactly 6 times.

Solution:

Let E_i be the event that the number i appears exactly 6 times ($i \in \{1, 2, 3, 4, 5, 6\}$). Since $6 + 6 = 12 > 10$, it is impossible for two different numbers to both appear exactly 6 times in 10 rolls. Thus, the events E_i are mutually exclusive.

$$P\left(\bigcup_{i=1}^6 E_i\right) = \sum_{i=1}^6 P(E_i) = 6 \times \left[\binom{10}{6} \left(\frac{1}{6} \right)^6 \left(\frac{5}{6} \right)^4 \right]$$

$$P = 6 \times 210 \left(\frac{1}{6} \right)^6 \left(\frac{5}{6} \right)^4 \approx 0.0125$$

4. You have 16 marbles: 3 blue, 4 green, and 9 red. You also have 3 jars. For each of the 16 marbles, you select a jar at random and place the marble into it. Assume that the jars are large enough to accommodate any number of marbles in them.

- (a) What is the probability that no jar is empty?

Solution:

We use the Principle of Inclusion-Exclusion. Let E_i be the event that jar i is empty (for $i = 1, 2, 3$). We want $1 - P(E_1 \cup E_2 \cup E_3)$. $P(E_i) = \left(\frac{2}{3}\right)^{16}$ $P(E_i \cap E_j) = \left(\frac{1}{3}\right)^{16}$ $P(E_1 \cap E_2 \cap E_3) = 0$

$$P(\text{no jar empty}) = 1 - \left(\binom{3}{1} \left(\frac{2}{3}\right)^{16} - \binom{3}{2} \left(\frac{1}{3}\right)^{16} \right) = 1 - 3 \left(\frac{2}{3}\right)^{16} + 3 \left(\frac{1}{3}\right)^{16} \approx 0.9954$$

- (b) What is the probability that each jar contains 3 red marbles in it?

Solution:

There are 9 red marbles in total. Because the jar selection for each marble is independent, we only need to consider the placements of the 9 red marbles. We want exactly 3 red marbles in Jar 1, 3 in Jar 2, and 3 in Jar 3. This is a multinomial probability:

$$P(3 \text{ red in each jar}) = \frac{9!}{3!3!3!} \left(\frac{1}{3}\right)^3 \left(\frac{1}{3}\right)^3 \left(\frac{1}{3}\right)^3 = \frac{1680}{3^9} = \frac{1680}{19683} \approx 0.0854$$