Class Exercise

1. **Agentic swarm**

   Masterbot manages 22 AI agents residing in my laptop.

   (a) In how many ways could the agents be assigned into four group projects requiring 9,5,4 and 4 agents respectively. Each agent is assigned to precisely one team.

   Solution:
   Assuming the group projects are distinct, this is a partition problem. We use the multinomial coefficient to find the number of ways to divide the 22 agents:

   $$\frac{22!}{9!5!4!4!}$$

   (b) In how many ways could the very same assignment be done if two of the AI agents absolutely refuse to be on the same team?

   Solution:
   We calculate the total number of unrestricted assignments and subtract the number of assignments where the two rival agents are on the same team. We find the combinations for the remaining 20 agents:

   - If both are on the team of 9: $\frac{20!}{7!5!4!4!}$
   - If both are on the team of 5: $\frac{20!}{9!3!4!4!}$
   - If both are on the first team of 4: $\frac{20!}{9!5!2!4!}$
   - If both are on the second team of 4: $\frac{20!}{9!5!4!2!}$

   The total number of valid assignments is the total minus these possibilities:

   $$\frac{22!}{9!5!4!4!} - \left(\frac{20!}{7!5!4!4!} + \frac{20!}{9!3!4!4!} + 2\left(\frac{20!}{9!5!2!4!}\right)\right)$$

2. **Poker hand**

   Two Pairs is a poker hand in which two of the cards are of the same rank, another two cards are also of the same rank, which however is different from the rank of the first pair and the fifth card is of a rank different from the ranks of the two pairs. Here is an example: $5\diamondsuit, 5\spadesuit, Q\clubsuit, Q\spadesuit, 10\diamondsuit$. What is the probability that a randomly selected poker hand is Two Pairs?

   Solution:
   The total number of possible 5-card hands from a standard 52-card deck is $\binom{52}{5}$.
   To form a Two Pairs hand, we must:

   - Choose the 2 ranks for the pairs: $\binom{13}{2}$
   - Choose the 2 suits for each of those ranks: $\binom{4}{2}\binom{4}{2}$
   - Choose 1 rank for the fifth card from the remaining 11 ranks: $\binom{11}{1}$

- Choose 1 suit for the fifth card: $\binom{4}{1}$

The probability is the product of these choices over the total possible hands:

$$P(\text{Two Pairs}) = \frac{\binom{13}{2}\binom{4}{2}^2\binom{11}{1}\binom{4}{1}}{\binom{52}{5}} = \frac{78 \times 36 \times 11 \times 4}{2598960} = \frac{123552}{2598960} \approx 0.0475$$

3. **Koala thieves**

   In 2006, thieves planning to steal a koala from a zoo in Australia, had to change their minds after it proved too vicious to be abducted. The perps, later confessed that their original plan was to swap the koala for drugs; but after it scratched hell out of them, they opted to steal a 1.4 metre crocodile instead - which they claimed was a lot easier[1].

   (a) A witness reported that a car seen speeding away from the zoo had a number plate that began with a V or W, the digits 4,7, and 8, and the end letters A,C, and E. However, they could not remember the order of the digits or the end letters. How many cars would need to be checked to be sure of including the suspect car?

   Solution:
   Based on the witness description, the license plate structure is [Start Letter] - [3 Digits] - [3 End Letters].

   - Choices for the starting letter (V or W): 2
   - Arrangements for the 3 distinct digits (4, 7, 8): $3! = 6$
   - Arrangements for the 3 distinct end letters (A, C, E): $3! = 6$

   Total possible cars to check $= 2 \times 6 \times 6 = 72$.

   (b) In other parts of Australia, a licence plate consists of a sequence of seven symbols: number, letter, letter, letter, number, number, number, where a letter is any one of 26 letters (A - Z) and a number is one of $(0-9)$. Assume that all licence plates are equally likely.

   i. What is the probability that all symbols are different?

   Solution:
   The total number of possible plates is $10 \times 26^3 \times 10^3 = 10^4 \times 26^3$.
   If all symbols are different, we are choosing and permuting 4 distinct digits out of 10, and 3 distinct letters out of 26: $P(10,4) \times P(26,3)$.

$$P(\text{all different}) = \frac{P(10,4) \times P(26,3)}{10^4 \times 26^3} = \frac{5040 \times 15600}{175760000} \approx 0.4474$$

   ii. What is the probability that all symbols are different and the first number is the largest among the numbers?

   Solution:
   If the four numbers are all different, exactly one of them is the largest. By symmetry, the largest number is equally likely to be in any of the 4 designated number

[1] https://www.smh.com.au/national/cranky-koala-meaner-than-stolen-croc-20060330-gdn9jt.html

positions. Therefore, the probability that the largest number occupies the very first position is $\frac{1}{4}$.

$$P(\text{first is largest} \mid \text{all different}) = \frac{1}{4}$$

$$P(\text{all different AND first is largest}) = \frac{1}{4} \times \frac{P(10,4) \times P(26,3)}{10^4 \times 26^3} \approx 0.1118$$

4. **Scandinavians**

A group of 18 Scandinavians consists of 5 Norwegians, 6 Swedes, and 7 Finns. They are seated at random in a row of chairs. Compute the following probabilities:

(a) that all the Norwegians sit together,

Solution:
Treat the 5 Norwegians as a single block. We are arranging this 1 block along with the remaining 13 people ($1 + 13 = 14$ items). The Norwegians can also be arranged among themselves within their block.

$$P = \frac{14! \times 5!}{18!}$$

(b) that all the Norwegians and all the Swedes sit together

Solution:
Treat the 5 Norwegians as 1 block and the 6 Swedes as 1 block. We are arranging these 2 blocks along with the remaining 7 Finns ($2 + 7 = 9$ items). Both groups can be arranged internally.

$$P = \frac{9! \times 5! \times 6!}{18!}$$

(c) that all the Norwegians, all the Swedes, and all the Finns sit together.

Solution:
Treat each nationality as a block (3 blocks total). We arrange the 3 blocks, and then arrange the individuals within their respective blocks.

$$P = \frac{3! \times 5! \times 6! \times 7!}{18!}$$

5. **Sampling a Chemical Process**

Hourly samples of a chemical process are collected for 24 samples per day, but due to the cost considerations only 5 samples are actually analysed each day. Asume the decision on which 5 samlpes are to be analysed is made at random.

a) In how many ways can the 5 analysed samples be distributed through the day if the the 5 analyses done are all different?

Solution:
Since the 5 analyses are different, the order in which we assign the samples to the analyses matters. We choose and arrange 5 samples out of 24.

$$P(24,5) = \frac{24!}{19!} = 24 \times 23 \times 22 \times 21 \times 20 = 5100480$$

b) In how many ways can the 5 analysed samples be distributed through the day if the 5 analyses done are identical?
Solution:
Since the analyses are identical, the order does not matter. We simply choose a combination of 5 samples out of 24.

$$\binom{24}{5} = \frac{24!}{5!19!} = 42504$$

c) What is the probability that all 5 analysed samples would be from the samples collected before noon if the analyses done are all different?
Solution:
There are 12 samples collected before noon. The number of successful outcomes is choosing and permuting 5 samples out of those 12.

$$P = \frac{P(12,5)}{P(24,5)} = \frac{12 \times 11 \times 10 \times 9 \times 8}{24 \times 23 \times 22 \times 21 \times 20} = \frac{95040}{5100480} \approx 0.0186$$

d) What is the probability that all 5 analysed samples would be from the samples collected before noon if the analyses done are identical?
Solution:
The probability remains the same whether the analyses are identical or distinct because the scaling factor of 5! applies equally to both the numerator and the denominator.

$$P = \frac{\binom{12}{5}}{\binom{24}{5}} = \frac{792}{42504} \approx 0.0186$$