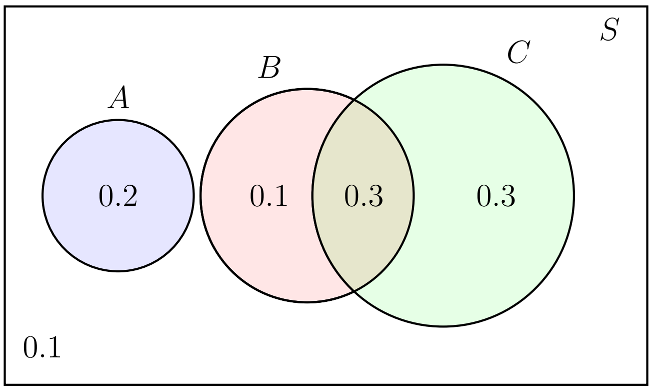
Class Exercise

1. **Three events**

Let A, B and C be three events such that A is mutually exclusive from both B and C, $P(A) = 0.2$, $P(B) = 0.4$, $P(C) = 0.6$ and $P(B \cap C) = 0.3$. Determine the following probabilities.



Solution:
Because A is mutually exclusive from B and C, we know $P(A \cap B) = 0$ and $P(A \cap C) = 0$. Using the given probabilities, the regions are populated as shown in the Venn diagram above.

(a) $P(A \cup B \cup C)$
$P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C) = 0.2 + 0.4 + 0.6 - 0 - 0 - 0.3 + 0 = \mathbf{0.9}$

(b) $P(A' \cap B \cap C)$
Since A is disjoint from B and C, the intersection $B \cap C$ is entirely contained within A'. Thus, $P(A' \cap B \cap C) = P(B \cap C) = \mathbf{0.3}$.

(c) $P(A' \cap B' \cap C')$
By De Morgan's Law, this is the probability of the complement of the union: $P((A \cup B \cup C)') = 1 - P(A \cup B \cup C) = 1 - 0.9 = \mathbf{0.1}$.

(d) $P((A \cup B) \cap C)$
Distributing the intersection: $P((A \cap C) \cup (B \cap C)) = P(\emptyset \cup (B \cap C)) = P(B \cap C) = \mathbf{0.3}$.

(e) $P((A \cup B') \cap C')$
Distributing the intersection: $P((A \cap C') \cup (B' \cap C'))$. Because $A \cap C = \emptyset$, all of A is inside C', so $A \cap C' = A$. Also, $B' \cap C' = (B \cup C)'$. Thus we want $P(A \cup (B \cup C)') = P(A) + P((B \cup C)') = 0.2 + (1 - (0.4 + 0.6 - 0.3)) = 0.2 + (1 - 0.7) = 0.2 + 0.3 = \mathbf{0.3}$.

(f) $P(C' \cup (A' \cap B'))$
Using De Morgan's Law on the second term: $P(C' \cup (A \cup B)') = P((C \cap (A \cup B))') = 1 - P(C \cap (A \cup B))$. From part (d), $P(C \cap (A \cup B)) = 0.3$, so $1 - 0.3 = \mathbf{0.7}$.

(g) $P(A \cup B' \cup C')$
Using De Morgan's Law: $P(A \cup (B \cap C)') = P((A' \cap (B \cap C))')$. Since A is mutually exclusive from B and C, $A' \cap (B \cap C) = B \cap C$. Therefore, $1 - P(B \cap C) = 1 - 0.3 = \mathbf{0.7}$.

(h) $P((A \cup B) \cap (A' \cup C))$
Expanding: $(A \cap A') \cup (A \cap C) \cup (B \cap A') \cup (B \cap C) = \emptyset \cup \emptyset \cup (B \cap A') \cup (B \cap C)$. Since A and B are mutually exclusive, $B \cap A' = B$. The expression simplifies to $B \cup (B \cap C) = B$. Thus, $P(B) = \mathbf{0.4}$.

2. **Dr. Newbee**
Phyllis feels numbness and tingling in some of the digits on her left foot. She visits her newly minted family physician, Dr. Newbee, and presents the symptoms. Dr. Newbee googles the symptoms and finds that two not mutually exclusive conditions, A and B, are possible. The prevalence of these conditions in the general population is as follows: $P(A) = 0.45$, $P(B) = 0.39$ and $P(A \cap B) = 0.04$. 36% of patients with condition A only show the symptoms, 90% of patients with condition B only show the symptoms, 99% of the patients afflicted by both conditions show the symptoms and 12% of patients who are not afflicted with either of the two conditions have these symptoms.

Solution Setup:
Let S be the event of presenting symptoms. First, we define the partition of the population:

- $P(A \cap B') = P(A) - P(A \cap B) = 0.45 - 0.04 = 0.41$
- $P(A' \cap B) = P(B) - P(A \cap B) = 0.39 - 0.04 = 0.35$
- $P(A \cap B) = 0.04$
- $P(A' \cap B') = 1 - (0.41 + 0.35 + 0.04) = 1 - 0.80 = 0.20$

Given the conditional probabilities: $P(S|A \cap B') = 0.36$, $P(S|A' \cap B) = 0.90$, $P(S|A \cap B) = 0.99$, and $P(S|A' \cap B') = 0.12$.

The total probability of showing symptoms S is:

$$\begin{aligned} P(S) &= P(S|A \cap B')P(A \cap B') + P(S|A' \cap B)P(A' \cap B) + P(S|A \cap B)P(A \cap B) + P(S|A' \cap B')P(A' \cap \\ &= (0.36)(0.41) + (0.90)(0.35) + (0.99)(0.04) + (0.12)(0.20) \\ &= 0.1476 + 0.3150 + 0.0396 + 0.0240 \\ &= 0.5262 \end{aligned}$$

(a) What is the probability that Phyllis suffers from only condition A?
We want to find $P(A \cap B'|S)$ using Bayes' Theorem:

$$P(A \cap B'|S) = \frac{P(S|A \cap B')P(A \cap B')}{P(S)} = \frac{0.1476}{0.5262} \approx \mathbf{0.2805}$$

(b) What is the probability that Phyllis is not afflicted with either of the two conditions A, B despite the symptoms?
We want to find $P(A' \cap B'|S)$:

$$P(A' \cap B'|S) = \frac{P(S|A' \cap B')P(A' \cap B')}{P(S)} = \frac{0.0240}{0.5262} \approx \mathbf{0.0456}$$

(c) Determine the probability that a randomly chosen person from the population is not afflicted by both conditions, given that he or she is afflicted by at least one of the two conditions.
Let E_1 be the event "not afflicted by both" i.e., $(A \cap B)'$. Let E_2 be the event "afflicted by at least one" i.e., $A \cup B$.

$$\begin{aligned} P((A \cap B)' | A \cup B) &= \frac{P((A \cap B)' \cap (A \cup B))}{P(A \cup B)} \\ &= \frac{P(A \cup B) - P(A \cap B)}{P(A \cup B)} \\ &= \frac{(0.45 + 0.39 - 0.04) - 0.04}{0.45 + 0.39 - 0.04} \\ &= \frac{0.80 - 0.04}{0.80} = \frac{0.76}{0.80} = \mathbf{0.95} \end{aligned}$$