Class Exercise

1. **Independence**
   If A and B are independent events prove that A' and B' are also independent.

   Solution:
   By the definition of independent events, $P(A \cap B) = P(A)P(B)$. We need to show that $P(A' \cap B') = P(A')P(B')$. Using De Morgan's laws and the basic axioms of probability:

$$
\begin{aligned}
P(A' \cap B') &= P((A \cup B)') \\
&= 1 - P(A \cup B) \\
&= 1 - (P(A) + P(B) - P(A \cap B)) \\
&= 1 - P(A) - P(B) + P(A)P(B) \\
&= (1 - P(A)) - P(B)(1 - P(A)) \\
&= (1 - P(A))(1 - P(B)) \\
&= P(A')P(B')
\end{aligned}
$$

   Thus, A' and B' are also independent.

2. What form does Bayes' rule for the posterior $p(A|B)$ take if A and B are independent events?

   Solution:
   Bayes' rule states that:

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

   If A and B are independent, then the probability of B occurring is unaffected by A, meaning $P(B|A) = P(B)$. Substituting this into Bayes' rule:

$$P(A|B) = \frac{P(B)P(A)}{P(B)} = P(A)$$

   Therefore, if the events are independent, the posterior probability $P(A|B)$ is simply equal to the prior probability $P(A)$.

3. A group of girls at a school are taking Advanced Cantonese which come in two modules: C1 and C2. Each girl takes only module C1, or only module C2, or both C1 and C2. The probability that a girl is taking C2 given that she is taking C1 is 0.2. The probability that a girl is taking C1 given that she is taking C2 is 0.33

   Find the probability that a girl selected at random

   Solution Setup:
   Let $P(C_1)$ and $P(C_2)$ be the probabilities of taking module C1 and C2, respectively. Since every girl takes at least one module, $P(C_1 \cup C_2) = 1$. We are given two conditional probabilities: 1) $P(C_2|C_1) = \frac{P(C_1 \cap C_2)}{P(C_1)} = 0.2 \implies P(C_1) = 5P(C_1 \cap C_2)$ 2) $P(C_1|C_2) = \frac{P(C_1 \cap C_2)}{P(C_2)} = 0.33 \implies P(C_2) = \frac{1}{0.33} P(C_1 \cap C_2) = \frac{100}{33} P(C_1 \cap C_2)$

Using the addition rule for probabilities:

$$\begin{aligned}
P(C_1 \cup C_2) &= P(C_1) + P(C_2) - P(C_1 \cap C_2) \\
1 &= 5P(C_1 \cap C_2) + \frac{100}{33}P(C_1 \cap C_2) - P(C_1 \cap C_2) \\
1 &= \left(4 + \frac{100}{33}\right) P(C_1 \cap C_2) \\
1 &= \frac{232}{33}P(C_1 \cap C_2) \\
P(C_1 \cap C_2) &= \frac{33}{232}
\end{aligned}$$

(a) is taking both C1 and C2.

Solution:
This is exactly the intersection $P(C_1 \cap C_2)$.

$$P(C_1 \cap C_2) = \frac{33}{232} \approx 0.1422$$

(b) is taking only C1.

Solution:
Taking only C1 corresponds to $P(C_1 \cap C_2')$, which is $P(C_1) - P(C_1 \cap C_2)$.

$$P(C_1 \cap C_2') = 5\left(\frac{33}{232}\right) - \frac{33}{232} = 4\left(\frac{33}{232}\right) = \frac{132}{232} = \frac{33}{58} \approx 0.5690$$

4. Roll a fair die 10 times. Compute the probability that you get

(a) at least one 6.

Solution:
It is easier to find the complement (rolling no 6s in 10 rolls).

$$P(\text{at least one 6}) = 1 - P(\text{no 6}) = 1 - \left(\frac{5}{6}\right)^{10} \approx 0.8385$$

(b) at least one 6 and at least one 5.

Solution:
Let A be the event of rolling no 6s, and B be the event of rolling no 5s. We want $1 - P(A \cup B)$. Using the addition rule: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$. $P(A) = \left(\frac{5}{6}\right)^{10}$ $P(B) = \left(\frac{5}{6}\right)^{10}$ $P(A \cap B)$ is the probability of rolling neither a 5 nor a 6: $\left(\frac{4}{6}\right)^{10}$

$$P(\text{at least one 6 and at least one 5}) = 1 - \left[2\left(\frac{5}{6}\right)^{10} - \left(\frac{4}{6}\right)^{10}\right] \approx 0.6943$$

(c) three 1's, two 2's, and five 3's.

Solution:
This follows a multinomial distribution:

$$P(3 \text{ 1s, } 2 \text{ 2s, } 5 \text{ 3s}) = \frac{10!}{3!2!5!}\left(\frac{1}{6}\right)^3\left(\frac{1}{6}\right)^2\left(\frac{1}{6}\right)^5 = 2520\left(\frac{1}{6}\right)^{10} \approx 0.0000417$$

(d) at least one number occurring exactly 6 times.

Solution:
Let E_i be the event that the number i appears exactly 6 times ($i \in \{1, 2, 3, 4, 5, 6\}$). Since $6+6 = 12 > 10$, it is impossible for two different numbers to both appear exactly 6 times in 10 rolls. Thus, the events E_i are mutually exclusive.

$$P\left(\bigcup_{i=1}^{6} E_i\right) = \sum_{i=1}^{6} P(E_i) = 6 \times \left[\binom{10}{6}\left(\frac{1}{6}\right)^6\left(\frac{5}{6}\right)^4\right]$$

$$P = 6 \times 210\left(\frac{1}{6}\right)^6\left(\frac{5}{6}\right)^4 \approx 0.0125$$

5. You have 16 marbles: 3 blue, 4 green, and 9 red. You also have 3 jars. For each of the 16 marbles, you select a jar at random and place the marble into it. Assume that the jars are large enough to accommodate any number of marbles in them.

(a) What is the probability that no jar is empty?

Solution:
We use the Principle of Inclusion-Exclusion. Let E_i be the event that jar i is empty (for $i = 1, 2, 3$). We want $1 - P(E_1 \cup E_2 \cup E_3)$. $P(E_i) = \left(\frac{2}{3}\right)^{16}$ $P(E_i \cap E_j) = \left(\frac{1}{3}\right)^{16}$ $P(E_1 \cap E_2 \cap E_3) = 0$

$$P(\text{no jar empty}) = 1 - \left(\binom{3}{1}\left(\frac{2}{3}\right)^{16} - \binom{3}{2}\left(\frac{1}{3}\right)^{16}\right) = 1 - 3\left(\frac{2}{3}\right)^{16} + 3\left(\frac{1}{3}\right)^{16} \approx 0.9954$$

(b) What is the probability that each jar contains 3 red marbles in it?

Solution:
There are 9 red marbles in total. Because the jar selection for each marble is independent, we only need to consider the placements of the 9 red marbles. We want exactly 3 red marbles in Jar 1, 3 in Jar 2, and 3 in Jar 3. This is a multinomial probability:

$$P(\text{3 red in each jar}) = \frac{9!}{3!3!3!}\left(\frac{1}{3}\right)^3\left(\frac{1}{3}\right)^3\left(\frac{1}{3}\right)^3 = \frac{1680}{3^9} = \frac{1680}{19683} \approx 0.0854$$