

Class Exercise

1. Rankings

Five men and five women are ranked according to their scores on an exam. Assume that no two scores are the same and all possible rankings are equally likely. Let the random variable X be the highest ranking achieved by a women. What is the probability mass function of X ? Organize your solution into a table.

Solution: There are $\binom{10}{5} = 252$ total equally likely ways to choose the 5 ranking positions for the women. If $X = k$, it means the highest-ranked woman is in the k -th position. For this to happen, the previous $k - 1$ positions must be occupied by men, and the remaining 4 women must be distributed among the remaining $10 - k$ positions. The number of ways this can occur is $\binom{10-k}{4}$. Since there are only 5 men, k can be at most 6.

- $P(X = 1) = \frac{\binom{9}{4}}{252} = \frac{126}{252} = \frac{1}{2}$
- $P(X = 2) = \frac{\binom{8}{4}}{252} = \frac{70}{252} = \frac{5}{18}$
- $P(X = 3) = \frac{\binom{7}{4}}{252} = \frac{35}{252} = \frac{5}{36}$
- $P(X = 4) = \frac{\binom{6}{4}}{252} = \frac{15}{252} = \frac{5}{84}$
- $P(X = 5) = \frac{\binom{5}{4}}{252} = \frac{5}{252}$
- $P(X = 6) = \frac{\binom{4}{4}}{252} = \frac{1}{252}$

Alternative Solution:

Instead of combinations, we can use permutations by considering the exact arrangement of all 10 people in the rankings. There are $10!$ total possible arrangements. If the highest-ranking woman is in position x , then the first $x - 1$ positions must be occupied by men. We arrange $x - 1$ men from the 5 available, pick 1 of the 5 women for position x , and arrange the remaining $10 - x$ people in the remaining spots.

- $P(X = 1) = \frac{{}^5P_1 \cdot 9!}{10!} = \frac{1}{2}$
- $P(X = 2) = \frac{{}^5P_1 \cdot {}^5P_1 \cdot 8!}{10!} = \frac{5}{18}$
- $P(X = 3) = \frac{{}^5P_2 \cdot {}^5P_1 \cdot 7!}{10!} = \frac{5}{36}$
- $P(X = 4) = \frac{{}^5P_3 \cdot {}^5P_1 \cdot 6!}{10!} = \frac{5}{84}$
- $P(X = 5) = \frac{{}^5P_4 \cdot {}^5P_1 \cdot 5!}{10!} = \frac{5}{252}$
- $P(X = 6) = \frac{{}^5P_5 \cdot {}^5P_1 \cdot 4!}{10!} = \frac{1}{252}$

x	1	2	3	4	5	6
$P(X = x)$	$\frac{1}{2}$	$\frac{5}{18}$	$\frac{5}{36}$	$\frac{5}{84}$	$\frac{5}{252}$	$\frac{1}{252}$

2. Lottery

A state has a 5/35 lottery game. The ticket costs 1\$. Matching all 5 numbers (jackpot) pays 200,000\$, matching 4 of the 5 numbers pays 200\$ and matching 3 of the 5 numbers pays 2\$. The rest of the tickets are non-winning. What is the expected value (in \$) of this game and what is the standard deviation?

Solution: The total number of possible combinations is $\binom{35}{5} = 324,632$. Let Y be the net profit of a single ticket. We find the probabilities of each prize and subtract the \$1 ticket cost from the payouts to get the net values:

- **Match 5:** $\binom{5}{5} = 1$ way. Net: \$199,999. Probability: $\frac{1}{324,632}$
- **Match 4:** $\binom{5}{4}\binom{30}{1} = 150$ ways. Net: \$199. Probability: $\frac{150}{324,632}$
- **Match 3:** $\binom{5}{3}\binom{30}{2} = 4,350$ ways. Net: \$1. Probability: $\frac{4,350}{324,632}$
- **Match <3:** All remaining combinations = 320,131 ways. Net: -\$1. Probability: $\frac{320,131}{324,632}$

Expected Value:

$$\begin{aligned} E(Y) &= \frac{1(199999) + 150(199) + 4350(1) + 320131(-1)}{324632} \\ &= \frac{199999 + 29850 + 4350 - 320131}{324632} = \frac{-85932}{324632} \approx -\$0.26 \end{aligned}$$

Standard Deviation: First find $E(Y^2)$:

$$\begin{aligned} E(Y^2) &= \frac{1(199999)^2 + 150(199)^2 + 4350(1)^2 + 320131(-1)^2}{324632} \\ &= \frac{39999600001 + 5940150 + 4350 + 320131}{324632} \approx 123234.50 \end{aligned}$$

$$\begin{aligned} Var(Y) &= E(Y^2) - [E(Y)]^2 = 123234.50 - (-0.2647)^2 \approx 123234.43 \\ \sigma(Y) &= \sqrt{123234.43} \approx \$351.05 \end{aligned}$$

3. Shifts

Show that $E(a + kX) = a + kE(X)$ and $Var(a + kX) = k^2Var(X)$ for any discrete random variable X and constants a and k .

Solution: By the definition of the expected value for a discrete random variable:

$$E(a + kX) = \sum_x (a + kx)P(X = x)$$

Distributing the terms inside the summation:

$$E(a + kX) = \sum_x (aP(X = x) + kxP(X = x))$$

Using the linearity of summation, we can split this into two sums and pull out the constants a and k :

$$E(a + kX) = a \sum_x P(X = x) + k \sum_x xP(X = x)$$

Since the sum of all probabilities over the sample space is 1 (meaning $\sum_x P(X = x) = 1$) and the definition of expected value is $\sum_x xP(X = x) = E(X)$, we substitute these in:

$$E(a + kX) = a(1) + kE(X) = a + kE(X)$$

By the definition of variance, $Var(Y) = E[(Y - E(Y))^2]$. Let our new variable be $Y = a + kX$. First, we substitute our earlier result for the expected value, $E(a + kX) = a + kE(X)$:

$$Var(a + kX) = E[((a + kX) - (a + kE(X)))^2]$$

Distributing the negative sign and simplifying the expression inside:

$$Var(a + kX) = E[(a + kX - a - kE(X))^2] = E[(kX - kE(X))^2]$$

Factoring out the constant k from the terms inside the square:

$$Var(a + kX) = E[(k(X - E(X)))^2] = k^2 E[(X - E(X))^2]$$

Since $E[(X - E(X))^2]$ is the exact definition of $Var(X)$:

$$Var(a + kX) = k^2 Var(X)$$

4. Chemicals

A chemical supply company currently has in stock 50kg of a certain chemical, which it sells to customers in 5kg containers. Let X be the number of containers ordered by a randomly chosen customer. Suppose that X has pmf

x	1	2	3	4
$p(x)$	0.2	0.4	0.3	0.1

- a) Compute $E(X)$, $Var(X)$ and $\sigma(X)$.
- b) Compute the expected number of kg's of chemical left after the next customer's order is shipped and the standard deviation of the number of kg's left.

Solution: a)

$$E(X) = 1(0.2) + 2(0.4) + 3(0.3) + 4(0.1) = 0.2 + 0.8 + 0.9 + 0.4 = 2.3$$

$$E(X^2) = 1^2(0.2) + 2^2(0.4) + 3^2(0.3) + 4^2(0.1) = 0.2 + 1.6 + 2.7 + 1.6 = 6.1$$

$$Var(X) = E(X^2) - [E(X)]^2 = 6.1 - (2.3)^2 = 6.1 - 5.29 = 0.81$$

$$\sigma(X) = \sqrt{0.81} = 0.9$$

- b) Let Y be the amount of chemical left in stock. Since they start with 50 kg and each container is 5 kg, $Y = 50 - 5X$.

$$E(Y) = E(50 - 5X) = 50 - 5E(X) = 50 - 5(2.3) = 38.5 \text{ kg}$$

$$\sigma(Y) = \sigma(50 - 5X) = |-5|\sigma(X) = 5(0.9) = 4.5 \text{ kg}$$

5. ***n* coins**

We toss n coins and each one shows heads with probability p , independently of each others. Each coin which shows heads is tossed again. What is the probability mass function of the number of heads resulting from the second round of tosses?

Solution: For any single coin to result in "heads" during the second round, it must satisfy two independent events consecutively: 1. It must show heads in the first round (probability p). 2. It must show heads again in the second round (probability p).

The overall probability of success (getting heads on the second toss) for a single coin is $p \cdot p = p^2$. Since the n coins are tossed independently, the number of heads in the second round, let's call it Y , follows a binomial distribution with parameters n and p^2 . Thus, $Y \sim \text{Bin}(n, p^2)$.

The probability mass function is:

$$P(Y = k) = \binom{n}{k} (p^2)^k (1 - p^2)^{n-k} \quad \text{for } k = 0, 1, 2, \dots, n$$

6. **Speeding**

Highway Patrol stops one of every twelve drivers who drive over the speed limit in the Airport Tunnel on HW13. In the early afternoon 750 drivers go through the Tunnel in a quarter hour and 45 of those drivers are over the speed limit.

a) What is the expected number of drivers stopped in a quarter hour and what is the standard deviation?

b) What is the probability that less than 3 drivers will be stopped in a quarter hour?

Solution: Let X be the number of drivers stopped. The total number of speeders is $n = 45$, and the probability of a speeder being stopped is $p = \frac{1}{12}$. X follows a binomial distribution: $X \sim \text{Bin}(n = 45, p = 1/12)$.

a) Expected number and standard deviation:

$$E(X) = np = 45 \left(\frac{1}{12} \right) = 3.75$$

$$\sigma(X) = \sqrt{np(1-p)} = \sqrt{45 \left(\frac{1}{12} \right) \left(\frac{11}{12} \right)} = \sqrt{3.4375} \approx 1.854$$

b) The probability that less than 3 drivers will be stopped is $P(X < 3) = P(X = 0) + P(X = 1) + P(X = 2)$:

$$P(X = 0) = \binom{45}{0} \left(\frac{1}{12} \right)^0 \left(\frac{11}{12} \right)^{45} \approx 0.0199$$

$$P(X = 1) = \binom{45}{1} \left(\frac{1}{12} \right)^1 \left(\frac{11}{12} \right)^{44} \approx 0.0978$$

$$P(X = 2) = \binom{45}{2} \left(\frac{1}{12} \right)^2 \left(\frac{11}{12} \right)^{43} \approx 0.2355$$

$$P(X < 3) \approx 0.0199 + 0.0978 + 0.2355 = 0.3532$$

The probability is approximately 35.32%.