# L9. The Binomial Distribution

**Example 1**

Some American gyms don't like Gen Z because they actually go to the gym regularly rather than buying a membership and rarely using it. Since the gyms are always busy, they have trouble attracting and retaining new customers.

A gym manager knows from experience that 60% of Gen Z members come in on any given weekday. Suppose there are 5 Gen Z members at a particular location.

a. What is the probability that exactly 2 of them show up today?

b. What is the probability that exactly 3 or 5 of them show up today?

c. What is the probability that none of them show up today?

d. How many Gen Z members are expected to show up today? What is the standard deviation?

**Solution**

Let $X$ = the number of Gen Z members who show up at the gym today.

Then $X$ is a binomial random variable with $n = 5$ and $p = 0.6$.

a. Probability that exactly 2 show up:

$$\begin{aligned} P(X = 2) &= C_2^5(0.6)^2(0.4)^3 \\ &= 0.2304 \end{aligned}$$

b. Probability that exactly 3 or 5 show up:

$$\begin{aligned} P(X = 3 \cup X = 5) &= P(X = 3) + P(X = 5) \\ &= C_3^5(0.6)^3(0.4)^2 + C_5^5(0.6)^5(0.4)^0 \\ &= 0.3456 + 0.07776 \\ &= 0.42336 \end{aligned}$$

c. Probability that none show up:

$$\begin{aligned} P(X = 0) &= C_0^5(0.6)^0(0.4)^5 \\ &= 0.01024 \end{aligned}$$

d. Expected value and standard deviation:

$$\begin{aligned}\mathbb{E}[X] &= np \\ &= 5 \cdot 0.6 \\ &= 3\end{aligned}$$

$$\begin{aligned}\mathbb{S}[X] &= \sqrt{np(1-p)} \\ &= \sqrt{5 \cdot 0.6 \cdot 0.4} \\ &= \sqrt{1.2} \approx 1.095\end{aligned}$$

**Example 2**

In June 1982, Mount Galunggung in Indonesia erupted, sending a massive cloud of volcanic ash into the atmosphere. British Airways Flight 9, en route from London to Auckland, unknowingly flew straight into it. The result: all four engines failed mid-flight.

In what has since become a legendary display of British composure and a masterpiece of understatement, the pilot addressed the passengers with this calmly delivered gem: "Ladies and gentlemen, this is your captain speaking. We have a small problem. All four engines have stopped. We are doing our damnedest to get them going again. I trust you are not in too much distress."

The probability that an engine, clogged with volcanic ash will restart working is 0.97. Assuming that all four engines on Flight 9 operate independently, determine the following.

a. What is the probability that all four engines will restart after being clogged with ash?

b. What is the probability that two of the engines will restart?

c. What is the probability that at least one of the engines not restart?

d. In order to execute a safe landing, at least three of the four the engines have to be working. What is the probability that the plane will land safely?

e. If $X$ is the number of engines that restart, what is the mean and standard deviation for this distribution?

**Solution**

Let $X$ = the number of engines that restart after being clogged with volcanic ash.

Then $X$ is a binomial random variable with $n = 4$ and $p = 0.97$.

a. Probability that all four engines restart:

$$\begin{aligned} P(X=4) &= C_4^4(0.97)^4(0.03)^0 \\ &= (0.97)^4 \\ &\approx 0.8853 \end{aligned}$$

b. Probability that exactly two engines restart:

$$\begin{aligned} P(X=2) &= C_2^4(0.97)^2(0.03)^2 \\ &= 6\cdot(0.9409)\cdot(0.0009) \\ &\approx 0.0051 \end{aligned}$$

c. Probability that at least one engine does not restart:

$$\begin{aligned} P(\text{at least one fails}) &= 1-P(\text{all succeed}) \\ &= 1-P(X=4) \\ &= 1-0.8853 \\ &= 0.1147 \end{aligned}$$

**Alternate Solution:** Let $Y$ = the number of engines that do not restart. Then $n=4$ and $p=0.03$

$$\begin{aligned} P(\geq 1 \text{ engine doesn't restart}) &= P(Y=1)+P(Y=2)+P(Y=3)+P(Y=4) \\ &= 1-P(Y=0) \\ &= 1-C_0^4(0.03)^0(0.97)^4 \\ &= 1-0.08853 \\ &= 0.1147 \end{aligned}$$

d. Probability of safe landing (at least 3 engines working):

$$\begin{aligned} P(X\geq 3) &= P(X=3)+P(X=4) \\ &= C_3^4(0.97)^3(0.03)^1+C_4^4(0.97)^4(0.03)^0 \\ &= 0.1095+0.8853 \\ &= 0.9948 \end{aligned}$$

e. Expected value and standard deviation:

$$\begin{aligned} \mathbb{E}[X] = np &= 4\cdot 0.97 \\ &= 3.88 \end{aligned}$$

$$\begin{aligned} \mathbb{S}[X] = \sqrt{np(1-p)} &= \sqrt{4\cdot 0.97\cdot 0.03} \\ &= \sqrt{0.1164} \\ &\approx 0.3411 \end{aligned}$$

### Example 3

In 2022 the Italian city of Parma, and Finalnd's capital city, Helsinki, were fighting over who should host the prestigious European food agency. Civil discussions broke down when then Italian Prime Minister, Silvio Berlusconi, quipped at an EU Summit that the agency couldn't go to Helsinki because "the Finns don't even know what prosciutto is". The Finns responded the next day by running a full page ad in the country's largest newspaper which said:

> "Prosciutto is Ham. Now 1.2 million Finns know this. Is that enough for Berlusconi?"

At a grocery store in Helsinki, a survey revealed that 90% of Finns knew what prosciutto was. In the next 12 customers who enter the store,

a. what is the probability that exactly ten customers know what prosciutto is?

b. what is the probability that eight or nine customers know what prosciutto is?

c. what is the probability that one or two, do not know what prosciutto is?

d. what is the expected number of customers who know what prosciutto is?

e. what is the standard deviation?

### Solution

Let $X$ = the number of customers (out of 12) who know what prosciutto is.

Then $X$ is a binomial random variable with $n = 12$ and $p = 0.9$.

a. Probability that exactly 10 customers know:

$$\begin{aligned} P(X = 10) &= C^{12}_{10}(0.9)^{10}(0.1)^2 \\ &\approx 0.2300 \end{aligned}$$

b. Probability that 8 or 9 customers know:

$$\begin{aligned} P(X = 8 \cup X = 9) &= P(X = 8) + P(X = 9) \\ &= C^{12}_{8}(0.9)^{8}(0.1)^4 + C^{12}_{9}(0.9)^{9}(0.1)^3 \\ &= 0.0213 + 0.0852 \\ &= 0.1065 \end{aligned}$$

c. Probability that one or two do not know

Let $Y$ = the number of people who do not know what prosciutto is. Then $n = 12$ with $p = 0.1$

$$\begin{aligned} P(Y = 1 \cup Y = 2) &= P(Y = 1) + P(Y = 2) \\ &= C^{12}_{1}(0.1)^{1}(0.9)^{11} + C^{12}_{2}(0.1)^{2}(0.9)^{10} \\ &= 0.3766 + 0.2301 \\ &= 0.6067 \end{aligned}$$

d. Expected number of customers who know:

$$\mathbb{E}[X] = np = 12 \cdot 0.9 = 10.8$$

e. Standard deviation:

$$\begin{aligned}\mathbb{S}[X] &= \sqrt{np(1-p)} \\ &= \sqrt{12 \cdot 0.9 \cdot 0.1} = \sqrt{1.08} \approx 1.0392\end{aligned}$$

**Example 4**

During their first term in office, US Vice-President Mike Pence was generally seen to be more popular than Donald Trump - but not by much. A Gallup Poll at the time revealed that 44% of voters found Pence favourable; whereas Trump was only popular with 43% of voters. While this might be good news for Pence, the bad news is that the same poll found that 12% of Americans had no idea of who he was.

In the next 20 Americans that you encounter, what is

a. the probability that none of them know who Mike Pence is?

b. the probability that more than 17 people do not know who Mike Pence is?

c. the probability that at most three people know who Mike Pence is?

d. the probability that three or four of them found Pence favourable?

e. the probability that five or eight of them found Pence unfavourable?

**Solution**

a. Let $X$ = the number of Americans (out of 20) with the specified opinion or recognition.

$$P(X = 20) = C_{20}^{20}(0.12)^{20}(0.88)^{0} = (0.12)^{20} \approx 1.15 \times 10^{-18}$$

b. Probability that more than 17 people do not know who Mike Pence is:

$$\begin{aligned}P(X > 17) &= P(X = 18) + P(X = 19) + P(X = 20) \\ &\approx \text{negligible values (all less than } 10^{-14}) \\ &\approx 0\end{aligned}$$

c. Probability that at most three people know who Mike Pence is:

Let $Y$ = the number of people who know who Pence is

$$\begin{aligned} P(Y \le 3) &= P(Y=0) + P(Y=1) + P(Y=2) + P(Y=3) \\ &= C_0^{20}(0.12)^0(0.88)^{20} + C_1^{20}(0.12)^1(0.88)^{19} + C_2^{20}(0.12)^2(0.88)^{18} + C_3^{20}(0.12)^3(0.88)^{17} \\ &= 0.0682 + 0.1860 + 0.2488 + 0.2122 \\ &= 0.7152 \end{aligned}$$

d. Let $Z$ = the number of people who found Pence favourable; $n = 20$ and $p = 0.44$

$$\begin{aligned} P(X = 3 \cup X = 4) &= P(X=3) + P(X=4) \\ &= C_3^{20}(0.44)^3(0.56)^{17} + C_4^{20}(0.44)^4(0.56)^{16} \\ &\approx 0.0160 + 0.0416 \\ &= 0.0576 \end{aligned}$$

e. Let $U$ = the number of people who found Pence unfavourable; $n = 20$ and $p = 0.56$

$$\begin{aligned} P(U = 5 \cup U = 8) &= P(U=5) + P(U=8) \\ &= C_5^{20}(0.56)^5(0.44)^{15} + C_8^{20}(0.56)^8(0.44)^{12} \\ &= 0.0038 + 0.0641 \\ &= 0.1980 \end{aligned}$$

### Example 5

In a recent poll, 56% of Americans believe "Arabic numerals" should not be taught in schools - a bold stance considering those are, the numbers 0 through 9. Of the next 10 Americans that you meet,

a. What is the probability that at least one of them believes that Arabic Numerals should not be taught in school?

b. What is the probability that at more than three of them believes that Arabic Numerals should not be taught in school?

c. What is the probability that at exactly four of them believes that Arabic Numerals should be taught in school?

d. What is the probability that at less than half of them believes that Arabic Numerals should be taught in school?

### Solution

Let $X$ = the number of Americans who believe that Arabic numerals should not be taught in schools.

Then $X$ is a binomial random variable with $n = 10$ and $p = 0.56$.

a. Probability that at least one believes Arabic numerals should not be taught:

$$\begin{aligned} P(X \geq 1) &= 1 - P(X = 0) \\ &= 1 - C_0^{10}(0.56)^0(0.44)^{10} \\ &\approx 1 - 0.000057 \\ &= 0.999943 \end{aligned}$$

b. Probability that more than three believe Arabic numerals should not be taught:

$$\begin{aligned} P(X > 3) &= 1 - P(X \leq 3) \\ &= 1 - [P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3)] \\ &= 1 - \big[C_0^{10}(0.56)^0(0.44)^{10} + C_1^{10}(0.56)^1(0.44)^9 \\ &\qquad + C_2^{10}(0.56)^2(0.44)^8 + C_3^{10}(0.56)^3(0.44)^7\big] \\ &\approx 1 - (0.000057 + 0.00072 + 0.0044 + 0.0159) \\ &\approx 1 - 0.0211 \\ &= 0.9789 \end{aligned}$$

c. Probability that exactly 4 believe Arabic numerals should be taught:

Let $Y$ = the number of people who think that Arabic numerals should be taught; $n = 10$, and $p = 0.44$

$$\begin{aligned} P(Y = 4) &= C_4^{10}(0.44)^4(0.56)^6 \\ &= 0.0264 \end{aligned}$$

d. Probability that fewer than half believe Arabic numerals should be taught:

$$\begin{aligned} P(Y < 5) &= P(Y = 0) + P(Y = 1) + P(Y = 2) + P(Y = 3) + P(Y = 4) \\ &= C_0^{10}(0.44)^0(0.56)^{10} + C_1^{10}(0.44)^1(0.56)^9 \\ &\qquad + C_2^{10}(0.44)^2(0.56)^8 + C_3^{10}(0.44)^3(0.56)^7 + C_4^{10}(0.44)^4(0.56)^6 \\ &= 0.2304 + 0.1644 + 0.0811 + 0.0234 + 0.0032 \\ &= 0.5025 \end{aligned}$$