L16. Hypothesis Test on the Mean; Single Sample, Variance Known

Example 1

Consider a hypothesis test where $H_0 : \mu = 29$ and $H_1 : \mu \neq 29$. A random sample of 25 observations taken from a population produced a sample mean of 25.3. The population is normally distributed with $\sigma = 8$. At the 5% level of significance, is there enough evidence to reject the null hypothesis?

Solution

We are given: $n = 25 \quad \overline{x} = 25.3 \quad \sigma = 8$

Step 1: Set up the null hypothesis

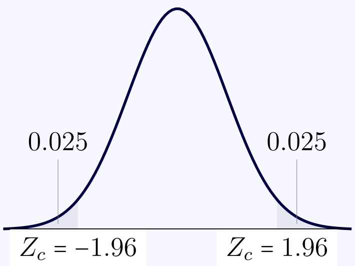
$$H_0 : \mu = 29$$

Step 2: Set up the alternate hypothesis

$$H_1 : \mu \neq 29$$ Double tail test.

Step 3: Using the level of significance and H_1, draw the rejection zone, and find the critical values of Z that sets the cut-off values for α

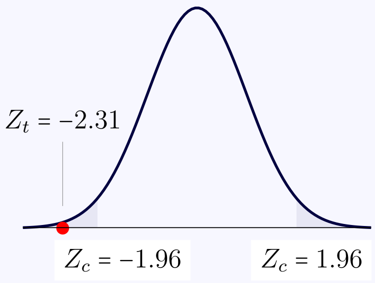
Since this is a double tail test, and the level of significance is $\alpha = 5\%$, we need to split the α into two. This means that there is 2.5% in each tail. Using the normal distribution to determine the critical values, Z_c, associated with these regions; we find that, $Z_{0.025} = \pm 1.96$.



Step 4: Using the data from the sample, calculate the test statistic.

$$Z_t = \frac{\overline{x} - k}{\sigma/\sqrt{n}} = \frac{25.3 - 29}{8/\sqrt{25}} = -2.31$$

Step 5: Make a statistical decision



$\because Z_t = -2.31$ is in the rejection zone $\Rightarrow$ Reject H_0

To do this without drawing a graph, we can simply compare the $Z-$ critical value against the Z_t, the test statistic.

$$|Z_c| = 1.96 < |Z_t| = 2.31 \quad \Rightarrow \quad \text{Reject } H_0.$$

Step 6: Compute the $P-$value to corroborate the statistical decision.

We need to compute the area that is blocked by our test statistic. Since this is a double tailed test, we have to double the area (since there are two areas (tails) of opportunity for where the test-statistic can fall into).

$$\begin{aligned} P-value = 2 \cdot P(Z < Z_t) = 2 \cdot P(Z < -2.31) &= 2(0.0104) \\ &= 0.0208 < \alpha = 0.05 \end{aligned}$$

Since the $P-value$ is smaller than the level of significance (5%) we reject H_0 in favour of H_1.

Step 7: Write a conclusion in the context of the problem.

At the 5% level of significance, there is sufficient evidence to indicate that the mean for this population is different from 29.

Example 2

Consider a hypothesis test where $H_0 : \mu = 30$ and $H_1 : \mu < 30$. A random sample of 36 observations taken from a population produced a sample mean of 27.6. The population has a standard deviation of $\sigma = 10$. At the 1% level of significance, is there enough evidence to reject the null hypothesis?

Solution

We are given: $n = 36 \quad \overline{x} = 27.6 \quad \sigma = 10$

Step 1: Set up the null hypothesis

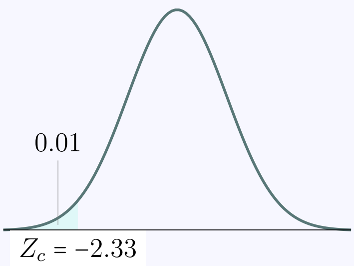
$$H_0 : \mu = 30$$

Step 2: Set up the alternate hypothesis

$$H_1 : \mu < 30$$ Left tail test.

Step 3: Using the level of significance and H_1, draw the rejection zone, and find the critical values of Z that sets the cut-off values for α

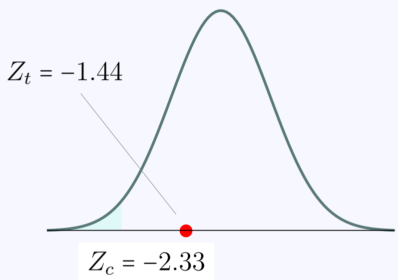
Since this is a left tail test, and the level of significance is $\alpha = 1\%$, we need to load all of the α into the left tail. Using the normal distribution to find the critical value, Z_c, associated with this region, we find that $Z_{0.01} = -2.33$.



Step 4: Using the data from the sample, calculate the test statistic.

$$Z_t = \frac{\overline{x} - k}{\sigma/\sqrt{n}} = \frac{27.6 - 30}{10/\sqrt{36}} = -1.44$$

Step 5: Make a statistical decision



$\because Z_t = -1.44$ is **not** in the rejection zone, $\Rightarrow$ fail to reject H_0.

To do this without drawing a graph, we can simply compare the $Z-$ critical value against the Z_t, the test statistic.

$$|Z_c| = 2.33 > |Z_t| = 1.44 \quad \Rightarrow \quad \text{Fail to Reject } H_0.$$

Step 6: Compute the $P-$value to corroborate the statistical decision.

We need to compute the area that is blocked by our test statistic. Since this only in one tail, there is no need to double the area (we only do this when it is a double tailed test).

$$P-value = P(Z < Z_t) = P(Z < -1.44) = 0.0749 > \alpha = 0.01$$

Since the $P-value$ is greater than the level of significance (1%) we fail to reject H_0

Step 7: Write a conclusion in the context of the problem.

At the 1% level of significance, there is insufficient evidence to indicate that the mean for this population is less than 30.

Example 3

Consider a hypothesis test where $H_0 : \mu = 54$ and $H_1 : \mu > 54$. A random sample of 40 observations taken from a population produced a sample mean of 56.78. The population has a standard deviation of $\sigma = 5.25$. At the 5% level of significance, is there enough evidence to reject the null hypothesis?

Solution

We are given: $n = 40$ $\quad \overline{x} = 56.78$ $\quad \sigma = 5.25$

$H_0 : \mu = 54$
$H_1 : \mu > 54$ Right tail test.

The Z_c, associated with this test is $Z_{0.05} = +1.645$

Calculation of the test statistic.

$$Z_t = \frac{\overline{x} - k}{\sigma/\sqrt{n}} = \frac{56.7 - 54}{5.25/\sqrt{40}} = 3.35$$

$$\because |Z_c| = 1.645 < |Z_t| = 3.35 \quad \Rightarrow \quad \text{Reject } H_0.$$

Calculation of $P-$value

$$\begin{aligned} P-value &= P(Z > Z_t) \\ &= P(Z > 3.35) = 0.0004 < \alpha = 0.05 \end{aligned}$$

$\because \quad P - value < \alpha \quad \Rightarrow \quad$ reject H_0

Conclusion:

At the 5% level of significance, there is sufficient evidence to indicate that the mean for this population is greater than 54.

Example 4

A certain colleague of mine who teaches Differential Equations suspects that the 10 ounce bag of fancy Swiss cheese he gets at the supermarket actually weighs less than 10 ounces. He took a random sample of 20 such packages and found that the mean weight for the sample was 9.955 ounces. The population follows a normal distribution with a standard deviation of 0.15 ounces.

At the $\alpha = 0.01$ level of significance does the data indicate that the average weight in this type of packaged cheese weighs less than 10 ounces? Compute a $P - value$ for your test and write a conclusion in the context of the problem.

Solution

We are given: $n = 20 \quad \overline{x} = 9.955$ ounces $\quad \sigma = 0.15$ ounces

$H_0 : \mu = 10$
$H_1 : \mu < 10$ Left tail test.

The Z_c, associated with this test is $Z_{0.01} = -2.33$

Calculation of the test statistic.

$$Z_t = \frac{\overline{x} - k}{\sigma/\sqrt{n}} = \frac{9.955 - 10}{0.15/\sqrt{20}} = -1.34$$

$$\because \quad |Z_c| = 2.33 > |Z_t| = 1.34 \quad \Rightarrow \quad \text{Fail to Reject } H_0.$$

Calculation of $P-$value

$$\begin{aligned} P - value &= P(Z < Z_t) \\ &= P(Z < -1.34) = 0.0901 > \alpha = 0.01 \end{aligned}$$

$\because \quad P - value > \alpha \quad \Rightarrow \quad$ Fail to Reject H_0

Conclusion:

At the 1% level of significance, there is insufficient evidence to indicate that the average weight of the packaged cheese is less than 10 ounces.

Example 5

A study claims that senior citizens living in Mirabel spend an average of 14 hours gardening during the weekend. A random sample of 200 people showed that these senior citizens spend an average of 14.65 hours on gardening during the weekend. Suppose that the standard deviation is known to be 3 hours.

At the 0.025 level of significance does the data indicate that the average amount of time spent on gardening by seniors living in Mirabel is more than 14 hours during the weekend? Compute a $P-value$ and write a conclusion in the context of the problem.

Solution

We are given: $n = 200$ $\overline{x} = 14.65$ hours $\sigma = 0.3$ hours

$H_0 : \mu = 14$
$H_1 : \mu > 14$ Right tail test.

The Z_c, associated with this test is $Z_{0.025} = 1.96$

Calculation of the test statistic.

$$Z_t = \frac{\overline{x} - k}{\sigma/\sqrt{n}} = \frac{14.65 - 14}{3/\sqrt{200}} = 3.06$$

$$\because \quad |Z_c| = 1.96 < |Z_t| = 3.06 \quad \Rightarrow \quad \text{Reject } H_0.$$

Calculation of $P-$value

$$\begin{aligned} P - value &= P(Z > Z_t) \\ &= P(Z > 3.06) = 0.0011 < \alpha = 0.025 \end{aligned}$$

$\because \quad P - value < \alpha \quad \Rightarrow \quad$ Reject H_0

Conclusion:

At the 2.5% level of significance, there is sufficient evidence to indicate that the average time spent on gardening by seniors living in Mirabel is greater than 14 hours.

Example 6

Humans are known to have a mean gestation period of 280 days with a standard deviation of about nine days. A hospital wondered whether there was any evidence that their patients were at risk for giving birth prematurely. In a random sample of 70 women, the average gestation time was 274.3 days. At the 0.005 level of significance does the data indicate that the average mean gestation period for women at this hospital is less than 280 days?

Solution

We are given: $n = 70$ $\quad \overline{x} = 274.3$ days $\quad \sigma = 0.9$ days

$$H_0 : \mu = 280$$
$$H_1 : \mu < 280 \qquad \text{Left tail test.}$$

The Z_c, associated with this test is $Z_{0.005} = -2.575$

Calculation of the test statistic.

$$Z_t = \frac{\overline{x} - k}{\sigma/\sqrt{n}} = \frac{274.3 - 280}{9/\sqrt{70}} = -5.30$$

$$\because \quad |Z_c| = 2.575 < |Z_t| = 5.30 \quad \Rightarrow \quad \text{Reject } H_0.$$

Calculation of P−value

$$\begin{aligned} P - value &= P(Z < Z_t) \\ &= P(Z < -5.30) = 0.0000 < \alpha = 0.005 \end{aligned}$$

$$\because \quad P - value < \alpha \quad \Rightarrow \quad \text{Reject } H_0$$

Conclusion:

At the 0.5% level of significance, there is sufficient evidence to indicate that the mean gestation period for women at this hospital is less than 280 days.

Example 7

The life in hours of a battery is known to be normally distributed with standard deviation $\sigma = 1.25$ hours. A random sample of 10 batteries has a mean life of $\overline{x} = 40.5$ hours.

a. Is there evidence to support the claim that battery life exceeds 40 hours? Use $\alpha = 0.05$. What is the $P - value$ for this test?

b. Construct an appropriate interval test to support the conclusion obtained in (a)

Solution

a. We have: $n = 10$ $\quad \overline{x} = 40.5$ hours $\quad \sigma = 1.25$ hours

$H_0 : \mu = 40$ Claimed mean life of batteries
$H_1 : \mu > 40$ Right tail

Test statistic: $Z_t = \dfrac{\overline{x} - k}{\sigma/\sqrt{n}} = \dfrac{40.5 - 40}{1.25/\sqrt{10}} = 1.26$

$$P - value = P(Z < Z_t) = P(Z > 1.26) = 0.1038 > \alpha = 0.05$$

$\Rightarrow$ Fail to reject H_0

At the 5% level of significance, there is insufficient evidence to indicate that the average life of these batteries is greater than 40 months.

b. LCB:

$$\begin{aligned} \overline{x} - Z_\alpha \cdot \frac{\sigma}{\sqrt{n}} &\le \mu \\ 40.5 - 1.645 \cdot \frac{1.25}{\sqrt{10}} &\le \mu \\ 39.85 &\le \mu \end{aligned}$$

Conclusion:

With repeated sampling, we are 95% confidence that the true lifespan of these bulbs is at least 39.85 days. Since 39.85 is less than 40 it does not support the a alternate hypothesis $\Rightarrow$ fail to reject H_0.

Example 8

An engineer who is studying the tensile strength of a steel alloy intended for use in golf club shafts knows that tensile strength is approximately normally distributed with $\sigma = 60$ psi. A random sample of 12 specimens has a mean tensile strength of $\overline{x} = 3450$ psi.

a. Test the hypothesis that mean strength is less than 3500 psi. Use $\alpha = 0.01$.

b. What is the smallest level of significance at which you would be willing to reject the null hypothesis?

c. Construct and explain how you could answer the question in part (a) with a one-sided confidence interval on mean tensile strength.

Solution

a. We have: $n = 12$ $\quad \overline{x} = 3450$ psi $\quad \sigma = 60$ psi

$H_0 : \mu = 3500$ Claimed mean tensile strength
$H_1 : \mu < 3500$ Left tail

Test statistic: $Z_t = \dfrac{\overline{x} - k}{\sigma/\sqrt{n}} = \dfrac{3450 - 3500}{60/\sqrt{12}} = -2.89$

$$P\text{-value} = P(Z < Z_t) = P(Z < -2.89) = 0.0019 < \alpha = 0.01$$

$\Rightarrow$ Reject H_0

At the 1% level of significance, there is sufficient evidence to indicate that the mean tensile strength is less than 3500 psi.

b. **Smallest level of significance**

The smallest level of significance at which we would reject H_0 is the P-value:

$$\alpha_{\min} = 0.0019$$

c. **UCB:**

$$\mu \leq \overline{x} + Z_\alpha \cdot \frac{\sigma}{\sqrt{n}}$$
$$\mu \leq 3450 + 2.33 \cdot \frac{60}{\sqrt{12}}$$
$$\mu \leq 3490.36$$

Conclusion:

We are 99% confident that the true mean tensile strength is at most 3490.36 psi. Since this maximum value is smaller than 3500, we reject H_0.

Example 9

At a dairy farm, a machine is set to fill 32-ounce cartons with milk. However, the machine does not put exactly 32-ounces into each carton; the amount varies from carton to carton but the volume is known to be normally distributed. When the machine is working correctly, the mean volume dispensed into each carton is 32 ounces, with a standard deviation of 1.5 ounces. A quality control inspector takes 25 cartons can finds that the average volume of milk in the containers is 31.93 ounces.

a. At the 5% level of significance, does the data indicate that the average amount of milk dispensed into the cartons by the machine is different from 32 ounces?

b. Construct and explain how a confidence interval can be used to support the conclusion obtained in (a).

Solution

a. We have: $n = 25 \quad \overline{x} = 31.93$ oz $\quad \sigma = 1.5$ oz

$H_0 : \mu = 32$ Claimed average fill volume
$H_1 : \mu \neq 32$ Two-tailed test

Test statistic: $Z_t = \dfrac{\overline{x} - k}{\sigma/\sqrt{n}} = \dfrac{31.93 - 32}{1.5/\sqrt{25}} = \dfrac{-0.07}{0.3} = -0.23$

$$P\text{-value} = 2 \cdot P(Z < Z_t) = 2 \cdot P(Z < -0.23) = 2 \cdot 0.4090 = 0.8180 > \alpha = 0.05$$

$\Rightarrow$ Fail to reject H_0

At the 5% level of significance, there is insufficient evidence to conclude that the average amount of milk dispensed is different from 32 ounces.

b. **CI:**

$$\overline{x} \pm Z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}}$$
$$31.93 \pm 1.96 \cdot \frac{1.5}{\sqrt{25}}$$
$$31.34 < \mu < 32.52$$

Conclusion:

With repeated sampling, we are 95% confident that the true mean amount of milk dispensed lies between 31.34 and 32.52 ounces. Since 32 is within this interval, it supports our decision to **fail to reject** H_0.