L18. Hypothesis Test on the Population Proportion

Example 1

A food company is planning to market a new type of frozen yoghurt. However, before marketing this yoghurt, the company wants to find the percentage of people who like it. The company's management has decided only to market this yoghurt if at least 35% of people like it. The company's research team selected a random sample of 400 people and asked then to taste this yoghurt. Of these, 112 said that they liked it. At the 2.5% level of significance, can you conclude that the company should market this yoghurt? What is the $P-value$?

Solution

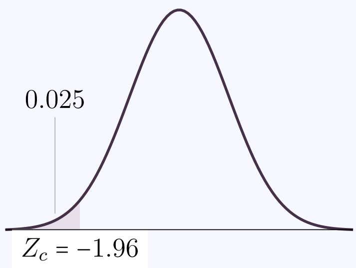
Let x = number of people who liked the product.

We have: $n = 400$ $\quad x = 112$ $\quad \hat{p} = \frac{x}{n} = \frac{112}{400} = 0.28$ $\quad \alpha = 0.025$

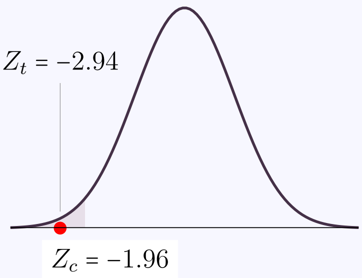
$H_0 : p = 0.35$ — Minimum percentage of likes in order to market
$H_1 : p < 0.35$ — Point estimate is less than 35%; left tail test

Check: $nk = 400(0.35) = 140 > 5$; $n(1-k) = 400(0.65) = 260 > 5$

If $\alpha = 0.025$ then



The test statistic is: $Z_t = \dfrac{\hat{p} - k}{\sqrt{\frac{k(1-k)}{n}}} = \dfrac{0.28 - 0.35}{\sqrt{\frac{0.35(1-0.35)}{400}}} = -2.94$



$\because \quad |Z_c| = 1.96 < |Z_t| = 2.94 \quad \Rightarrow \quad$ Reject H_0.

At the 2.5% level of significance, there is sufficient evidence to indicate that the percentage of people who like this yoghurt is less than 35%. The company should not market this yoghurt.

$P - value = P(Z < Z_t) = P(Z < -2.94) = 0.0016$

Example 2

A study in 2015 claimed that 11% of all children in the US currently live with at least on grandparent. In 2020, a random sample of 1600 children found that 180 did currently live with at least grandparent. At the 10% level of significance does the data indicate that the proportion of all children in the US who live with at least one grandparent is different from 11%? What is the $P - value$ for this test?

Solution

Let x = the number of children who live with at least one grandparent

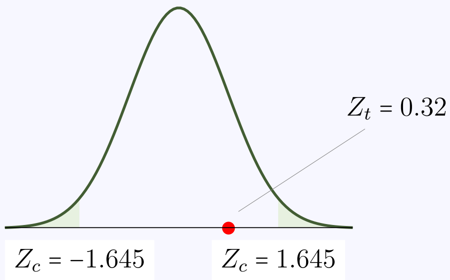
We have: $n = 1600 \quad x = 180 \quad \hat{p} = \frac{x}{n} = \frac{180}{1600} = 0.1125 \quad \alpha = 0.10$

$H_0 : p = 0.11$
$H_1 : p \neq 0.11$ Problem says "is different"

Check:
$nk = 1600(0.11) = 176 > 5 \quad ; \quad n(1-k) = 1600(0.89) = 1424 > 5$

If $\alpha = 0.10 \quad \Rightarrow \quad \frac{\alpha}{2} = 0.05 \quad \Rightarrow \quad Z_c = \pm 1.645$

Test statistic: $Z_t = \dfrac{\hat{p} - k}{\sqrt{\frac{k(1-k)}{n}}} = \dfrac{0.1125 - 0.11}{\sqrt{\frac{0.11(1-0.11)}{1600}}} = 0.32$



$\because \quad |Z_c| = 1.645 > |Z_t| = 0.32 \Rightarrow \quad$ Fail to reject H_0

At the 10% level of significance, there is insufficient evidence to indicate percentage of all children in the US who currently live with at least one grandparent is different from 11%.

$P - value = 2 \cdot P(Z > Z_t) = 2 \cdot P(Z > 0.32) = 2(0.3745) = 0.7490$

Example 3

A company that sell computer parts claims that more 90% of their orders are mailed within 72 hours of them being received. The quality control department took a random sample of 150 orders and found that 140 were mailed within 72 hours of the order being placed. At the 0.005 level of significance does the data indicate that the company's claim is true?

Solution

Let x = the number of orders mailed within 72 hours of being received.

We have: $n = 150 \quad x = 140 \quad \hat{p} = \frac{x}{n} = \frac{140}{150} = 0.9333 \quad \alpha = 0.005$

$H_0 : \mu = 0.90$
$H_1 : p > 0.90$

Check: $nk = 150(0.90) = 135 > 5 \quad ; \quad n(1-k) = 150(0.10) = 15 > 5$

If $\alpha = 0.005 \quad \Rightarrow \quad Z_c = +2.575$

Test statistic:

$$Z_t = \frac{\hat{p} - k}{\sqrt{\frac{k(1-k)}{n}}} = \frac{0.9333 - 0.90}{\sqrt{\frac{0.90(1-0.90)}{150}}} = 1.36$$

$$P - value = P(Z > Z_t) = P(Z > 1.36)$$
$$= 0.0869 > \alpha = 0.001 \quad \Rightarrow \quad \text{Fail to reject } H_0$$

At the 0.5% level of significance, there is insufficient evident to support the company's claim that more than 90% of their orders are mailed within 72 hours of being placed.

Example 4

A researcher claims that at least 10% of all football helmets have manufacturing flaws that could potentially cause injury to the wearer. A sample of 200 helmets revealed that 24 helmets contained such defects.

a. Does this finding support the researcher's claim? Use $\alpha = 0.01$. Find the P-value.

b. Explain and show how the question in part (a) could be answered with a confidence bound.

Solution

a. We have: $n = 200$ $\quad \hat{p} = \frac{24}{200} = 0.12$

$H_0 : p = 0.10$ Claimed proportion with flaws
$H_1 : p > 0.10$ Right-tailed test

Test statistic:

$$Z_t = \frac{\hat{p} - k}{\sqrt{\frac{k(1-k)}{n}}} = \frac{0.12 - 0.10}{\sqrt{\frac{0.10 \cdot 0.90}{200}}} = 0.9434$$

$$P\text{-value} = P(Z > Z_t) = P(Z > 0.94) = 1 - 0.8276 = 0.1724 > \alpha = 0.01$$

$\Rightarrow$ Fail to reject H_0

At the 1% level of significance, there is insufficient evidence to support the researcher's claim that more than 10% of football helmets have manufacturing flaws.

b. **LCB:**

$$\hat{p} - Z_\alpha \cdot \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \leq p$$
$$0.12 - 2.33 \cdot \sqrt{\frac{0.12 \cdot 0.88}{200}} \leq p$$
$$0.0666 \leq p$$

Conclusion:

We are 99% confident that the true proportion of flawed helmets is at least 6.66%. Since 10% is not below the bound, this confidence interval does not support rejecting H_0. Thus, it agrees with the result of the hypothesis test.

Example 5

A random sample of 500 registered voters in Phoenix is asked if they favour the use of oxygenated fuels year-round to reduce air pollution. If more than 315 voters respond positively, we will conclude that at least 60% of the voters favour the use of these fuels.

a. What is the smallest level of significance for which the the null hypothesis will be rejected if exactly 60% of the voters favour the use of these fuels?

b. Construct and interpret an appropriate interval estimate to corroborate the conclusion obtained in (a).

Solution

a. We have: $n = 500$ $x = 316$ $\hat{p} = \frac{316}{500} = 0.632$ $p_0 = 0.60$

$H_0 : p = 0.60$ Proportion of voters in favor is 60%
$H_1 : p > 0.60$ Right-tailed test

Test statistic:

$$Z_t = \frac{\hat{p} - k}{\sqrt{\frac{k(1-k)}{n}}} = \frac{0.632 - 0.60}{\sqrt{\frac{0.60 \cdot 0.40}{500}}} = \frac{0.032}{0.0219} \approx 1.4612$$

$$\alpha_{min} = P(Z > 1.46) = 1 - P(Z < 1.46) = 0.0723$$

$\Rightarrow$ The smallest level of significance at which we would reject H_0 is approximately $\alpha = 0.0723$.

Conclusion:

The P-value for the test is approximately 0.0723.

- If $\alpha > 0.0723$ (e.g., $\alpha = 0.10$), then P-value $< \alpha \Rightarrow$ **Reject** H_0. There is sufficient evidence to conclude that more than 60% of voters favor the use of oxygenated fuels.
- If $\alpha < 0.0723$ (e.g., $\alpha = 0.05$), then P-value $> \alpha \Rightarrow$ **Fail to reject** H_0. There is insufficient evidence to support the claim that more than 60% of voters favor the use of oxygenated fuels.

b. **Case 1: (fail to reject H_0):**

Assume $\alpha = 0.05$, so the confidence level is 95% and $Z = 1.645$:

$$
\begin{aligned}
\hat{p} - Z_\alpha \cdot \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} &\leq p \\
0.632 - 1.645 \cdot \sqrt{\frac{0.632 \cdot 0.368}{500}} &\leq p \\
0.5965 &\leq p
\end{aligned}
$$

Interpretation:

We are 95% confident that the true proportion of voters who support the use of oxygenated fuels year-round is at least 59.65%. Since this value is **less than** 60%, the interval does **not** support the claim that more than 60% of voters are in favor. This agrees with the decision to **fail to reject** H_0.

Case 2: (We reject H_0):

Assume $\alpha = 0.10$, so the confidence level is 90% and $Z = 1.28$:

$$
\begin{aligned}
\hat{p} - Z_\alpha \cdot \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} &\leq p \\
0.632 - 1.28 \cdot \sqrt{\frac{0.632 \cdot 0.368}{500}} &\leq p \\
0.6044 &\leq p
\end{aligned}
$$

Interpretation:

We are 90% confident that the true proportion of voters who support the use of oxygenated fuels year-round is at least 60.44%. Since this value is **greater than** 60%, this supports the conclusion to **reject** H_0.