SN1 Practice Test 2

General Information and Recommendations

- Test 2 is scheduled to take place on **Wednesday, April 1st, 2026** in **E-509**.
- It covers lectures:
  - L5. Probability Theory (Introduction to Probability)
  - L6. Probability Theory (Counting Techniques)
  - L7. Probability Theory (Rules of Probability)
  - L8. Probability Mass Functions
  - L9. Expected Value and Variance of a Discrete Random Variable
  - L10. Binomial Distribution
- It is strongly advised that you go **over all of the problems covered in class, the class exercises, take home assignments and the problems on this test**.
- Solutions to this practice test will be posted by Monday, March 30 (9:00 am).

201-SN1-05 Probability and Statistics

Practice Test 2 - B

Winter 2025

Name:	Solutions

This test consists of 7 questions.
You will have **2 hours** to complete this test.

Instructions:

- Write your answers directly on the questionnaire.
- Show all work. Your solutions will be scored on the correctness and completeness of your methods and use of proper notation as well as your answers. A final answer with no work, calculations, and/or explanations will result in a grade of zero for that questions - even if it is correct.
- Notation counts. Poor notation = Loss of marks.
- All cell phones and listening devices must be turned off. All unauthorized materials must be put away.
- Only non-graphing, non-programmable calculators are permitted.
- Give exact answers and reduce all fractions. $\sqrt{2}$ is exact, 1.41 is an approximation of $\sqrt{2}$. If using decimals, please give answers to four significant decimal places.

Note:

- Some questions will take more time, some less. Manage your time.
- Start by reading over the entire test.
- Start with a question you find easy.

Marks	
1	/
2	/
3	/
4	/
5	/
6	/
7	/
Total: / (%)	

Good Luck!

Cheating and plagiarism are serious academic offences. Anyone caught cheating, or aiding in the act of cheating, will immediately be given a mark of zero for this test, and a note will be placed in his or her file.

1. **Ivan**

I have a colleague in the Math Department whose name is Ivan T. Ivanov. Suppose that the letters of his name IVANTIVANOV are arranged at random. What is the probability that

(a) the I's are together?

(b) the V's are not together?

(c) the letters A, N, and O appear together as a block, in that exact order (A-N-O)?

(d) the letters I-V-A-N appear together as a block, in that exact order?

Solution

The name `IVANTIVANOV` has 11 letters.

The total number of distinct arrangements is:

$$n(S) = \frac{11!}{2! \cdot 3! \cdot 2! \cdot 2!} = 831600$$

(a) Probability that the I's are together:

Treat both I's as a block `'II'` $\rightarrow$ reduces to 9 items.

$$n(\texttt{I's are toogether }) = \frac{9!}{2! \cdot 2! \cdot 3!} \times 10 \text{ positions } = 151200$$

$$P(\text{I's together}) = \frac{151200}{831600} = \frac{2}{11}$$

(b) Probability that the V's are not together:

Treat the three V's as a block `'VVV'` $\rightarrow$ reduces to 8 items but some of the items repeat.

$$n(\texttt{ V's are together}) = \frac{8!}{2! \cdot 2! \cdot 2!} \times 9 \text{ spots } = 45360$$

$$\begin{aligned} P(\texttt{V's not together}) &= 1 - P(\texttt{ V's are together}) \\ &= 1 - \frac{45360}{831600} \\ &= \frac{52}{55} \end{aligned}$$

(c) Probability that A, N, and O appear together as a block, in that exact order (A-N-O):

Treat the `ANO` as a block $\rightarrow$ reduces to 8 items, but some of the items repeat.

$$n(\texttt{ANO are together }) = \frac{8!}{2! \cdot 3!} \times 9 \text{ spots } = 30240$$

$$P(\texttt{ANO are together }) = \frac{30240}{831600} = \frac{2}{55}$$

(d) Probability that I-V-A-N appear together as a block in that exact order:

Strategy: We will treat and count the number of sequences that have (at least) one `IVAN` in it. If two `IVAN` blocks appear these cases will included in our counting.

$$n(\text{ at least one block of } \texttt{IVAN} \ ) = \frac{7!}{2!} \times 8 \text{ spots} = 20160$$

$$P(\text{ the at least one block of } \texttt{IVAN}) = \frac{20160}{831600} = \frac{4}{165}$$

2. **What's For Dinner?**

2013 is the first year in recorded history where consumption of fish has surpassed beef. In a survey, 100 people were asked if they ate fish, meat, both, or neither. You are told that 55 eat meat, 52 eat fish, and 21 eat neither. What is the probability that a randomly selected person

Solution

Let M = the person eats meat

F = the person eats fish

Draw a Venn diagram to help visualize the probabilities.

(a) eats either fish or meat?

$$\begin{aligned} P(M \cup F) &= 1 - P(F \cup M)' \\ &= 1 - \frac{21}{100} \\ &= \frac{79}{100} \end{aligned}$$

(b) eats both?

$$P(M \cap F) = \frac{28}{100}$$

(c) eats only meat?

$$\begin{aligned} P(M \cap F') &= P(M) - P(M \cap F) \\ &= \frac{55}{100} - \frac{28}{100} \\ &= \frac{27}{100} \end{aligned}$$

(d) doesn't eat fish?

$$\begin{aligned} P(F') &= 1 - P(F) \\ &= 1 - \frac{52}{100} \\ &= \frac{48}{100} \end{aligned}$$

(e) eats meat if they eat fish?

$$\begin{aligned} P(M \mid F) &= \frac{P(M \cap F)}{P(F)} \\ &= \frac{\frac{28}{100}}{\frac{52}{100}} \\ &= \frac{7}{13} \end{aligned}$$

3. **H & M**

A Swedish power plant is reducing its dependence on fossil fuels by burning clothes from H&M instead of coal.Some of these clothes are made of cotton, while others are made of synthetic materials. Cotton clothes burn more cleanly, whereas synthetic ones are more likely to produce pollution such as visible smoke.

Suppose that 60% of the clothing sent to the plant is synthetic, and the remaining 40% is cotton. When burned, 70% of the synthetic clothing produces visible smoke. In contrast, only 20% of the cotton clothing produces visible smoke.

(a) What is the probability that a piece of clothing produces visible smoke?

Solution

Let $S =$ synthetic clothing
$C =$ cotton clothing
$V =$ visible smoke

$$\begin{aligned} P(S) &= 0.6 \qquad & P(C) &= 0.4 \\ P(V \mid S) &= 0.7 \qquad & P(V \mid C) &= 0.2 \end{aligned}$$

Then

$$\begin{aligned} P(V) &= P(V|S) \cdot P(S) + P(V|C) \cdot P(C) \\ &= 0.6 \cdot 0.7 + 0.4 \cdot 0.2 \\ &= 0.50 \end{aligned}$$

(b) What is the probability that a piece of clothing produces visible smoke given that it is synthetic?

$$P(V|S) = 0.70$$

(c) If a piece of clothing produced does not produce visible smoke, what is the probability that it was synthetic?

Bayes' Theorem

$$P(S|V') = \frac{P(S \cap V')}{P(V')} = \frac{P(V'|S) \cdot P(S)}{1 - P(V)} = \frac{0.6 \cdot 0.3}{0.5} = 0.36$$

(d) What is the probability that a piece of clothing is either synthetic or produces visible smoke?

$$\begin{aligned} P(S \cup V) = P(S) + P(V) - P(S \cap V) &= P(S) + P(V) - P(V \mid S) \cdot P(S) \\ &= 0.6 + 0.5 - 0.6 \cdot 0.7 \\ &= 0.68 \end{aligned}$$

(e) What is the probability that a piece of clothing is either synthetic or produces visible smoke, but not both?

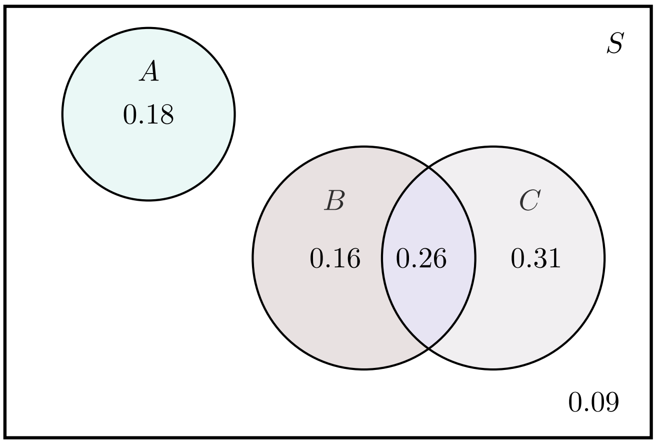
$$\begin{aligned} P(S \cup V) - P(S \cap V) &= 0.68 - 0.42 \\ &= 0.26 \end{aligned}$$

4. **Three Events**

Let A, B, and C be three events such that A is disjoint (mutually exclusive) from B and C. Suppose that $P(A) = 0.18$, $P(B) = 0.42$, $P(C) = 0.57$, and $P(B \cap C) = 0.26$. Compute

Solution

Venn Diagram



(a) $P\left((A \cup B') \cap (C \cup A')\right)$

Includes C and not B, and outside

$$P\left(\left(A \cup B'\right) \cap \left(C \cup A'\right)\right) = 0.31 + 0.09 = 0.40$$

(b) $P\left(A' \cap (B \cup C')\right)$

Includes $B \cap C$, B and not C, and outside

$$P\left(A' \cap \left(B \cup C'\right)\right) = 0.26 + 0.16 + 0.09 = 0.51$$

(c) $P\left((A \cup B \cup C')'\right)$

This is exactly C and not B

$$P\left(\left(A \cup B \cup C'\right)'\right) = 0.31$$

(d) $P\left((A \cup B') \cup (C \cap B)\right)$

This includes A, C and not B, $C \cap B$, and outside

$$P\left(\left(A \cup B'\right) \cup (C \cap B)\right) = 0.31 + 0.26 + 0.18 + 0.09 = 0.84$$

(e) $P\left((B' \cap C') \cup (A \cap (B \cup C)')\right)$

This includes A and outside

$$P\left(\left(B' \cap C'\right) \cup \left(A \cap (B \cup C)'\right)\right) = 0.18 + 0.09 = 0.27$$

5. **Game of Dice**

On a long train journey, a statistician is invited by a gambler to play a dice game. The game uses two ordinary dice which the statistician is to throw. If the total score is 12 , the statistician is paid \$6 by the gambler. If the total score is 8 , the statistician is to be paid \$3 by the gambler. However, if both or either dice shows a 1, the statistician pays the gambler \$2.

Let X be the amount paid to the statistician by the gambler after the dice are thrown once.

(a) Determine the probability mass function of X and organize the information into a table.

Solution

Two dice: $\Rightarrow$ $n(S) = 6 \times 6 = 36$ equally likely outcomes

Winnings (statistician receives money):

Total $= 12 \rightarrow +\$6$ (only outcome: $(6,6)$)

Total $= 8 \rightarrow +\$3$ (outcomes: $(2,6),(3,5),(4,4),(5,3),(6,2)) \rightarrow 5$ outcomes

Losses (either or both dice show a $1 \rightarrow$ pay \$2):

One or both dice show a 1 : outcomes include any roll with a 1 :

First die is 1 : $(1,1)$ to $(1,6) = 6$ outcomes

Second die is 1 : $(2,1)$ to $(6,1) = 5$ more (excluding $(1,1)$ already counted)

Total: $6+5=11$ outcomes $\rightarrow$ pay \$2

All other games; Total: 19 outcomes $\rightarrow$ pay \$0

Therefore the probability mass function of X is

X	6	3	-2	0
$P(X=x_i)$	$\frac{1}{36}$	$\frac{5}{36}$	$\frac{11}{36}$	$\frac{19}{36}$

(b) Determine the amount of money that the statistician is expected to win or lose after playing 100 games.

$$E(X) = \sum x_i \cdot P(X=x_i) = 6 \cdot \frac{1}{36} + 3 \cdot \frac{5}{36} + (-2) \cdot \frac{11}{36} + 0 \cdot \frac{19}{36}$$
$$= -\frac{1}{36}$$

Expected gain per game $= -\frac{1}{36}$ **Over 100 games:** $100 \cdot \left(-\frac{1}{36}\right) = -2.78.$

The statistician is expected to **lose approximately \$2.78** after 100 games.

(c) Determine the amount, k, that the 6 would have to be changed to in order to make the game unbiased (i.e. fair).

Let k be the unknown prize for a total of 12.

$$
\begin{aligned}
E(X) &= \sum x_i \cdot P(X = x_i) \\
0 &= k \cdot \frac{1}{36} + 3 \cdot \frac{5}{36} - 2 \cdot \frac{11}{36} + 0 \cdot \frac{19}{36} \\
0 &= \frac{k}{36} + \frac{15}{36} - \frac{22}{36} \\
k &= 7 \quad ;
\end{aligned}
$$

To make the game fair, the \$6 prize must be changed to **\$7**.

6. **Slot Machines**
Suppose that there are two slot machines, one of which pays out 10% of the time and the other pays out 20% of the time. Unfortunately, you have no idea which is which. Suppose you randomly choose a machine and put in a quarter.

(a) If you don't get a jackpot, what is the chance that you chose the machine that pays out 20% of the time?

Solution

Let $M_1 =$ machine that pays out 10%
$M_2 =$ machine that pays out 20%
$J =$you get a jackpot

Then

$$P(M_1) = P(M_2) = 0.5, \quad P(J|M_1) = 0.1, \quad P(J|M_2) = 0.2$$

Want $P(M_2|J')$; Bayes Theorem

$$P(M_2|J') = \frac{P(M_2 \cap J')}{P(J')} = \frac{P(J'|M_2)P(M_2)}{1 - P(J)}$$

$$\begin{aligned} P(J) &= P(J|M_1) \cdot P(M_1) + P(J|M_2) \cdot P(M_2) \\ &= (0.1) \cdot (0.5) + (0.2) \cdot (0.5) \\ &= 0.15 \end{aligned}$$

$$\begin{aligned} P(J') &= 1 - P(J) \\ &= 1 - 0.15 \\ &= 0.85 \end{aligned}$$

$$\therefore \quad P(M_2|J') = \frac{P(J'|M_2)P(M_2)}{1 - P(J)} = \frac{0.8 \cdot 0.5}{0.85} = 0.4706$$

(b) If you had instead gotten a jackpot, what would be the chance that you chose the one that pays out 20% of the time?

Bayes' Theorem

$$P(M_2|J) = \frac{M_2 \cap J}{P(J)} = \frac{P(J|M_2)P(M_2)}{P(J)} = \frac{0.2 \cdot 0.5}{0.15} = \frac{2}{3}$$

(c) What is the probability that you either chose the machine that pays out 20% of the time or you got a jackpot?

$$\begin{aligned} P(M_2 \cup J) &= P(M_2) + P(J) - P(M_2 \cap J) \\ &= P(M_2) + P(J) - P(J \mid M_2) \cdot P(M_2) \\ &= 0.5 + 0.15 - (0.5 \cdot 0.2) \\ &= 0.55 \end{aligned}$$

(d) Are the events "you chose the machine that pays out 20%" and "you got a jackpot" independent? Justify your answer with calculations.

Need to check if:

$$\begin{aligned} P(M_2 \cap J) &=^? P(M_2) \cdot P(J) \\ P(J|M_2) \cdot P(M_2) &=^? P(M_2) \cdot P(J) \\ 0.2 \cdot 0.5 &=^? (0.5) \cdot (0.15) \\ 0.1 &\neq 0.075 \end{aligned}$$

Since $0.1 \neq 0.075$, the events are **not independent.**

(e) Three people independently choose a slot machine at random (each with a 50% chance of picking either machine), and each plays once. What is the probability that all three chose the machine that pays out 10% of the time and all got jackpots?

$$\begin{aligned} P[(M_1 \cap J) \cap (M_1 \cap J) \cap (M_1 \cap J)] &= [P(M_1 \cap J)] \cdot [P(M_1 \cap J)] \cdot [P(M_1 \cap J)] \\ &= (0.5)^3 \cdot (0.1)^3 \\ &= 0.000125 \end{aligned}$$

7. **It's Greek to Me**

The equivalents of the English saying "That's Greek to me" are: "This appears to be Spanish" (German), "This is Russian to me" (Dutch), 'It's German to me" (Philippines), "It's Hebrew" (Finnish), "It's Chinese to me" (Hebrew), "Sounds like Mars language/These are chicken intestines" (China).

A group of girls at a school are taking Advanced Cantonese which come in two modules: C1 and C2. Each girl takes only module C1, or only module C2, or both C1 and C2. The probability that a girl is taking C2 given that she is taking C1 is 0.2. The probability that a girl is taking C1 given that she is taking C2 is 0.33

Find the probability that a girl selected at random

Solution

(a) is taking both C1 and C2.

Let C_1 = girl is taking module C_1
C_2 = girl is taking module C_2

$$P(C_2|C_1) = 0.20 \qquad P(C_1|C_2) = 0.33 \qquad P(C_1 \cup C_2) = 1$$

$$\begin{aligned} P(C_1 \cap C_2) &= P(C_2|C_1)P(C_1) \\ &= 0.2P(C_1) \end{aligned} \quad \begin{aligned} P(C_1 \cap C_2) &= P(C_1|C_2)P(C_2) \\ &= 0.33P(C_2) \end{aligned}$$

Since these two values must be the same:

$$0.2P(C_1) = 0.33P(C_2)$$

Isolating for, $P(C_1)$, we get

$$\Rightarrow P(C_1) = \frac{0.33P(C_2)}{0.2} = 1.65P(C_2)$$

As a result,

$$\begin{aligned} P(C_1 \cup C_2) &= P(C_1) + P(C_2) - P(C_1 \cap C_2) \\ 1 &= 1.65P(C_2) + P(C_2) - 0.33P(C_2) \\ 1 &= 2.23P(C_2) \\ 0.4310 &= P(C_2) \end{aligned}$$

$$\therefore P(C_1 \cap C_2) = 0.33P(C_2) = 0.33(0.4310) = 0.1422$$

(b) is taking only C1.

$$\begin{aligned} P(C_1 \cap C_2') = P(C_1) - P(C_1 \cap C_2) = 1.65P(C_2) - P(C_1 \cap C_2) &= 1.65(0.4310) - 0.1422 \\ &= 0.56895 \end{aligned}$$