SN1 Practice Test 3

General Information and Recommendations

- Test 3 is scheduled to take place on **Thursday, December 11**.
- It covers lectures:
  - L9. The Binomial Distribution
  - L10. The Normal Distribution
  - L11. The Normal Approximation to the Binomial
  - L12. Sampling Distributions and the Central Limit Theorem
  - L13. Confidence Interval on the Mean; Single Population, Variance Known
  - L14. Confidence Interval on the Mean; Single Population, Variance Unknown
  - L15. Confidence Interval on the Population Proportion
  - L16. Hypothesis Tests on the Mean; Single Population, Variance Known
  - L17. Hypothesis Tests on the Mean; Single Population, Variance Unknown
  - L18. Hypothesis Tests on the Population Proportion
- It is strongly advised that you go **over all of the problems covered in class**, the in class exercises, take home assignments, and the questions on the practice test.
- Solutions to this practice test will be posted by Wednesday, December 3 (9:00 pm).

201-SN1-05 Probability and Statistics

Practice Test 3 - A

Winter 2025

Name:

This test consists of 10 questions.
You will have **2 hours** to complete this test.

Instructions:

- Write your answers directly on the questionnaire.
- Show all work. Your solutions will be scored on the correctness and completeness of your methods and use of proper notation as well as your answers. A final answer with no work, calculations, and/or explanations will result in a grade of zero for that questions - even if it is correct.
- Notation counts. Poor notation = Loss of marks.
- All cell phones and listening devices must be turned off. All unauthorized materials must be put away.
- Only non-graphing, non-programmable calculators are permitted.
- Give exact answers and reduce all fractions. $\sqrt{2}$ is exact, 1.41 is an approximation of $\sqrt{2}$. If using decimals, please give answers to four significant decimal places.

Note:

- Some questions will take more time, some less. Manage your time.
- Start by reading over the entire test.
- Start with a question you find easy.

Marks	
1	/
2	/
3	/
4	/
5	/
6	/
7	/
8	/
9	/
10	/
Total: / (%)	

Good Luck!

Cheating and plagiarism are serious academic offences. Anyone caught cheating, or aiding in the act of cheating, will immediately be given a mark of zero for this test, and a note will be placed in his or her file.

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1. **Apples**

An apple a day does not keep the doctor away. A 2015 study found no correlation between eating apples and physician visits. However an apple a day is linked to fewer prescriptions.

The diameters of apples grown in an orchard are normally distributed with mean 3.4 and standard deviation 0.85 inches.
Solution: Let X be the diameter. $X \sim N(\mu = 3.4, \sigma = 0.85)$.

(a) What proportion of the apples has their diameters greater than 4 inches?

$$P(X > 4) = P\left(Z > \frac{4 - 3.4}{0.85}\right) = P(Z > 0.7059) \approx P(Z > 0.71)$$

Using the Z-table: $1 - 0.7611 = \mathbf{0.2389}$

(b) What is the probability that a randomly selected apple has diameters of between 2.5 and 4.5 inches?

$$P(2.5 < X < 4.5) = P\left(\frac{2.5 - 3.4}{0.85} < Z < \frac{4.5 - 3.4}{0.85}\right)$$

$$= P(-1.06 < Z < 1.29) = 0.9015 - 0.1446 = \mathbf{0.7569}$$

(c) 95% of the apples will have a diameter greater than what value? We need x such that $P(X > x) = 0.95$. This means the area to the left is 0.05. Looking up the Z-table for area 0.05, $Z \approx -1.645$.

$$x = \mu + Z\sigma = 3.4 + (-1.645)(0.85) = 3.4 - 1.398 = \mathbf{2.00} \text{ inches}$$

(d) Suppose that we take a random sample of eight apples from this orchard. What is the probability that the average diameter of the eight apples is smaller than 2.8 inches? $n = 8$. Standard error $\sigma_{\bar{x}} = \frac{0.85}{\sqrt{8}} \approx 0.3005$.

$$P(\bar{X} < 2.8) = P\left(Z < \frac{2.8 - 3.4}{0.3005}\right) = P(Z < -1.996) \approx P(Z < -2.00)$$

$$= \mathbf{0.0228}$$

2. **Coin Toss**

A coin is biased, coming up Heads $\frac{1}{5}$ of the times.
Solution: Let X = number of heads. $p = 0.2, q = 0.8$.

(a) You toss the coin 12 times. What is the probability that at less than a third of the tosses result in heads? $n = 12$. "Less than a third" means $< \frac{12}{3} = 4$. So we want $P(X < 4) = P(X \leq 3)$. Using binomial probability $P(X = k) = \binom{n}{k} p^k q^{n-k}$:

- $P(0) = \binom{12}{0}(0.2)^0(0.8)^{12} \approx 0.0687$
- $P(1) = \binom{12}{1}(0.2)^1(0.8)^{11} \approx 0.2062$
- $P(2) = \binom{12}{2}(0.2)^2(0.8)^{10} \approx 0.2835$
- $P(3) = \binom{12}{3}(0.2)^3(0.8)^9 \approx 0.2362$

Sum $\approx 0.0687 + 0.2062 + 0.2835 + 0.2362 = \mathbf{0.7946}$

(b) You toss the coin 10 times. What is the probability that at 8 or more of the tosses result in tails? "8 or more tails" is equivalent to "2 or fewer heads". $n = 10$. We want $P(X \leq 2)$.

- $P(0) = \binom{10}{0}(0.2)^0(0.8)^{10} \approx 0.1074$
- $P(1) = \binom{10}{1}(0.2)^1(0.8)^9 \approx 0.2684$
- $P(2) = \binom{10}{2}(0.2)^2(0.8)^8 \approx 0.3020$

Sum $\approx 0.1074 + 0.2684 + 0.3020 = \mathbf{0.6778}$

(c) Your toss the coin 1000 times. What is the probability that at least 220 of the coin tosses result in heads? $n = 1000, p = 0.2$. $\mu = np = 200$. $\sigma = \sqrt{npq} = \sqrt{1000(0.2)(0.8)} = \sqrt{160} \approx 12.65$. Using Normal Approximation with continuity correction ($P(X \geq 220) \to P(X \geq 219.5)$):

$$Z = \frac{219.5 - 200}{12.65} = 1.54$$

$$P(Z > 1.54) = 1 - 0.9382 = \mathbf{0.0618}$$

(d) Your toss the coin 200 times. What is the probability that between 135 and 215 (inclusive) of the coin tosses result in tails? Let Y = number of tails. $n = 200, p_{tails} = 0.8$. $\mu_Y = 160$. $\sigma_Y = \sqrt{200(0.8)(0.2)} = \sqrt{32} \approx 5.66$. We want $P(135 \leq Y \leq 215)$. Correct for continuity: $P(134.5 \leq Y \leq 215.5)$.

$$Z_1 = \frac{134.5 - 160}{5.66} = -4.50 \quad Z_2 = \frac{215.5 - 160}{5.66} = 9.80$$

Area is effectively **1** (or 0.9999).

3. **Yoghurt**

A company claims that there its 8-ounce low fat yoghurt has an average of 150 calories per cup. A consumer agency wanted to check the if the company's claim is true. A random sample of 10 such cups produced the following data on calories:

149	150	163	143	146	161	165	156	146	161

(a) Calculate the mean and the standard deviation for the data above. Include units in your answers.

$$\bar{x} = \frac{1540}{10} = \mathbf{154} \text{ cal}$$

$$s = \sqrt{\frac{\sum(x-\bar{x})^2}{n-1}} \approx \mathbf{8.37} \text{ cal}$$

(b) Using results from (a) formulate an appropriate hypothesis to test the company's claim each cup of yoghurt contains 150 calories. Use $\alpha = 0.05$ and estimate the $P-$value. $H_0 : \mu = 150$ vs $H_a : \mu \neq 150$. (Unknown variance $\rightarrow$ T-test).

$$t = \frac{\bar{x} - \mu_0}{s/\sqrt{n}} = \frac{154 - 150}{8.37/\sqrt{10}} = \frac{4}{2.647} \approx 1.51$$

$df = n - 1 = 9$. Looking at T-table for row 9: $t = 1.51$ falls between $t_{0.10} = 1.383$ and $t_{0.05} = 1.833$. The one-tail P-value is between 0.05 and 0.10. Since this is a two-tailed test, P-value is between 0.10 and 0.20. (Approx **0.165**). Since $P > 0.05$, we **fail to reject** H_0. There is insufficient evidence to say the mean is different from 150.

(c) Construct and explain how an appropriate confidence interval/bound could be used to support the conclusion of the test obtained in part (b). Construct a 95% Confidence Interval (two-sided). $t_{0.025,9} = 2.262$.

$$\bar{x} \pm t\left(\frac{s}{\sqrt{n}}\right) = 154 \pm 2.262(2.647) = 154 \pm 5.99$$

$$[148.01, 159.99]$$

Since the hypothesized value of 150 falls inside the interval, we cannot reject the null hypothesis, supporting the conclusion in (b).

4. For each question below, select **all** the statements that are **correct**. Each question has **at least one correct answer, but not necessarily all options are correct**. You will receive **full credit** if and only if you select all correct answers and **no incorrect answers**. Selecting an incorrect option or missing a correct option may result in **partial credit or no credit**.

   (a) A nutritionist claims that the average sodium content in a brand of soup is more than 600 mg. A random sample of 20 cans is taken, and the sample yields a mean sodium content of 622 mg with a sample standard deviation of 25 mg. A one-sided t-test is conducted with $H_0 : \mu = 600$ and $H_a : \mu > 600$, resulting in a p-value of 0.012. Assume a significance level of $\alpha = 0.05$. Which of the following statements are correct?

   X ☐ A t-test is appropriate because the population standard deviation is unknown.

   X ☐ The sample provides statistically significant evidence that the mean sodium content exceeds 600 mg.

   ☐ The p-value of 0.012 is greater than the significance level, so we fail to reject the null hypothesis.

   X ☐ The p-value indicates that if the true mean were 600 mg, there is a 1.2% chance of obtaining a sample mean of 622 mg or higher.

   ☐ Since the sample size is less than 30, no inference can be made about the population mean.

   (b) A company is testing a new feature in their app. The null hypothesis is H_0: the new feature has no effect on user engagement. The alternative hypothesis is H_a: the new feature increases user engagement. Which of the following statements are correct?

   X ☐ A Type I error would mean concluding that the feature increases engagement when it actually does not.

   X ☐ A Type II error would mean failing to detect an improvement in engagement when the feature actually does help.

   X ☐ A smaller significance level (α) decreases the chance of a Type I error.

   X ☐ The only way to reduce both Type I and Type II errors simultaneously is to increase the sample size.

   ☐ A Type II error is worse than a Type I error in every situation.

   (c) Suppose a researcher takes random samples of size $n = 5$ from a population with an unknown shape. They want to examine the sampling distribution of the sample mean $\bar{x}$. Which of the following statements are correct?

   X ☐ Since the sample size is very small, the shape of the population distribution greatly affects the shape of the sampling distribution.

   ☐ The Central Limit Theorem guarantees the sampling distribution of the mean will be approximately normal, regardless of sample size.

   X ☐ If the population is normally distributed, the sampling distribution of the sample mean will also be normal, even for small n.

   X ☐ With $n = 5$, we cannot assume the sampling distribution of the mean is approximately normal unless the population itself is normal.

   ☐ A sample size of 5 is always sufficient to assume normality of the sampling distribution.

(d) Suppose a population has an unknown distribution with a mean of $\mu = 100$ and a standard deviation of $\sigma = 20$. A researcher takes random samples of size $n = 50$ and computes the sample mean $\bar{x}$ for each sample. Which of the following statements are correct?

- X ☐ According to the Central Limit Theorem, the sampling distribution of $\bar{x}$ will be approximately normal.
- X ☐ The mean of the sampling distribution of $\bar{x}$ will be 100.
- X ☐ The standard deviation of the sampling distribution of $\bar{x}$ is $\frac{20}{\sqrt{50}}$.
- ☐ The Central Limit Theorem requires the population to be normally distributed, regardless of sample size.
- X ☐ The Central Limit Theorem becomes more accurate as the sample size increases.

5. **Textbooks**

An editor of a New York publishing company claims that the average time taken to write a textbook is 3 years. A random sample of 25 textbook authors found that the mean time taken for them to write a textbook was 38 months with a population standard deviation of 4.8 months.

(a) At the 1% level of significance, does the data indicate that the average time that it takes to write a textbook is more than 3 years? $H_0 : \mu = 36$ months vs $H_a : \mu > 36$ months. Given: $\sigma = 4.8, n = 25, \bar{x} = 38, \alpha = 0.01$. Since σ is known, we use Z-test.

$$Z = \frac{38 - 36}{4.8/\sqrt{25}} = \frac{2}{0.96} = 2.083$$

P-value: $P(Z > 2.08) = 1 - 0.9812 = \mathbf{0.0188}$. Since $0.0188 > 0.01$, we **fail to reject** H_0. There is not enough evidence at the 1% level.

(b) Construct and explain how appropriate confidence interval/bound can be used to support the conclusion of the test obtained in part (a). Since H_a is "greater than", we check the Lower Confidence Bound (99% confidence). $Z_{0.01} = 2.33$.

$$\mu > \bar{x} - Z_\alpha \frac{\sigma}{\sqrt{n}} = 38 - 2.33(0.96) = 38 - 2.24 = 35.76$$

The range of plausible values is $(35.76, \infty)$. Since the null value (36) is in this range, we cannot reject it.

(c) State the Type I and Type II errors in the context of the problem. **Type I:** Concluding that the average time is more than 3 years when it is actually 3 years. **Type II:** Concluding that the average time is 3 years when it is actually more than 3 years.

6. **Restaurants**

Some restaurant owners in China have tried an unusual method to get customers to keep coming back. At least 215 restaurants have been caught lacing their noodles with opiates to addict their customers.

Authorities estimate that no more than 5% of restaurants use illegal additives to boost repeat customers. A journalist collects a random sample of 400 restaurants and finds that 36 of them had positive tests for opiate substances in their food.

(a) At the 2% level of significance, does the data indicate that more than 5% of restaurants are using opiates. What is the $P-value$ for this test? $H_0 : p \leq 0.05$ vs $H_a : p > 0.05$. $\hat{p} = \frac{36}{400} = 0.09$.

$$Z = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0 q_0}{n}}} = \frac{0.09 - 0.05}{\sqrt{\frac{0.05(0.95)}{400}}} = \frac{0.04}{0.0109} = 3.67$$

$P(Z > 3.67) \approx 0.0001$. Since $P < 0.02$, we **reject** H_0. Yes, the data indicates more than 5%.

(b) Construct and explain how appropriate confidence interval/bound can be used to support the conclusion of the test obtained in part (a). 98% Lower Confidence Bound (since test is one-sided upper). $Z_{0.02} = 2.054$.

$$LB = \hat{p} - Z\sqrt{\frac{\hat{p}\hat{q}}{n}} = 0.09 - 2.054\sqrt{\frac{0.09(0.91)}{400}}$$

$$LB = 0.09 - 2.054(0.0143) = 0.09 - 0.029 = 0.061$$

The interval is $(0.061, 1]$. Since 0.05 is NOT in the interval, we reject H_0.

(c) A health inspector wants to estimate the true proportion of restaurants using opiates with a margin of error no greater than 3% at the 98% level of confidence. A preliminary investigation suggests the proportion may be around 7.5%. What is the minimum sample size the inspector should use to meet this level of precision? $E = 0.03$. $CL = 98\% \rightarrow \alpha = 0.02 \rightarrow Z_{\alpha/2} = Z_{0.01} = 2.33$. $\hat{p} = 0.075$.

$$n = \hat{p}(1 - \hat{p})\left(\frac{Z}{E}\right)^2 = 0.075(0.925)\left(\frac{2.33}{0.03}\right)^2$$

$$n = 0.069375(6032.11) = 418.47$$

Round up to **419**.

7. **Electric Fish**

The Romans used shocks from electric fish to treat gout, headaches, hysteria, and paralysis. A historian claims that the average duration of these ancient treatments was more than 45 minutes. A sample of 50 documented treatments was examined, and the mean duration was found to be 44.2 minutes. The population standard deviation is known to be 5.1 minutes.

(a) At the 5% level of significance, does the data support the claim that the average treatment duration was less than 45 minutes? What is the p-value for this test? (Note: The historian claimed MORE than 45, but the question asks to test if it was LESS). $H_0 : \mu = 45$ vs $H_a : \mu < 45$. $\sigma = 5.1$ (known) $\rightarrow$ Z-test.

$$Z = \frac{44.2 - 45}{5.1/\sqrt{50}} = \frac{-0.8}{0.721} = -1.11$$

P-value $= P(Z < -1.11) = \mathbf{0.1335}$. Since $0.1335 > 0.05$, **fail to reject** H_0. Data does not significantly support "less than 45".

(b) Construct and explain how an appropriate confidence interval or bound can be used to support the conclusion of the test in part (a). 95% Upper Confidence Bound (testing "less than"). $Z_{0.05} = 1.645$.

$$UB = \bar{x} + Z\frac{\sigma}{\sqrt{n}} = 44.2 + 1.645(0.721) = 44.2 + 1.186 = 45.39$$

Interval $(-\infty, 45.39]$. Since 45 is in the interval, we cannot reject the null.

(c) A researcher wants to estimate the average duration of electric fish treatments to within 1.5 minutes, with 95% confidence. What is the minimum sample size they should collect to meet this level of precision? $E = 1.5, Z_{0.025} = 1.96, \sigma = 5.1$.

$$n = \left(\frac{Z\sigma}{E}\right)^2 = \left(\frac{1.96 \cdot 5.1}{1.5}\right)^2 = (6.664)^2 = 44.4$$

Round up to **45**.

8. **Feather Fever**

In the 19th century, hat feathers were so popular that a single heron feather could cost as much as twice its weight in gold. Fashion trends and trade records indicate that the weights of feathers used in luxury hats were approximately normally distributed, with a mean of 22 grams and a standard deviation of 4 grams.

(a) What is the probability that a randomly selected feather weighs between 18.75 grams and 27.56 grams?

$$P(18.75 < X < 27.56) = P\left(\frac{18.75 - 22}{4} < Z < \frac{27.56 - 22}{4}\right)$$

$$= P(-0.8125 < Z < 1.39) \approx P(-0.81 < Z < 1.39)$$

$$= 0.9177 - 0.2090 = \mathbf{0.7087}$$

(b) What is the probability that out of 10 randomly selected feathers, at least 3 weigh more than 26 grams? First find $p = P(X > 26)$.

$$p = P\left(Z > \frac{26 - 22}{4}\right) = P(Z > 1.00) = 1 - 0.8413 = 0.1587$$

Now let $Y \sim Bin(n = 10, p = 0.1587)$. Find $P(Y \geq 3) = 1 - P(Y \leq 2)$.

- $P(0) = (0.8413)^{10} \approx 0.177$
- $P(1) = 10(0.1587)(0.8413)^9 \approx 0.335$
- $P(2) = 45(0.1587)^2(0.8413)^8 \approx 0.284$

Sum ≈ 0.796. $P(Y \geq 3) = 1 - 0.796 = \mathbf{0.204}$

(c) If 36 feathers are randomly selected, what is the probability that their average weight is less than 21 grams? $n = 36$. $\sigma_{\bar{x}} = 4/\sqrt{36} = 0.667$.

$$P(\bar{X} < 21) = P\left(Z < \frac{21 - 22}{0.667}\right) = P(Z < -1.50) = \mathbf{0.0668}$$

(d) Suppose the mean weight remains 22 grams. What value of the standard deviation of the sample mean $\sigma_{\bar{x}}$ would ensure that 95% of all sample means fall between 16.2 grams and 27.8 grams? The interval is symmetric around 22 (22 ± 5.8). For 95%, we need $1.96 \cdot \sigma_{\bar{x}} =$ Margin.

$$1.96\sigma_{\bar{x}} = 5.8$$

$$\sigma_{\bar{x}} = \frac{5.8}{1.96} \approx \mathbf{2.96} \text{ grams}$$

9. **Mountain Dew**

In 2009, a man tried to sue PepsiCo after he allegedly found a mouse in his can of Mountain Dew. Lawyers for the soft drink giant, refuted the man's claim by stating that Mountain Dew *could* dissolve a mouse in 30 days, and showed that his can was purchased 74 days after the container had been sealed. The case was settled out of court.

Scientists for PepsiCo, tested 36 samples of Mountain Dew and found that it took an average of 32 days with a standard deviation of 5 days for the beverage to dissolve a mouse completely. Assume that the time its takes for soda to dissolve a mouse is normally distributed.

(a) Construct a two-sided 99% confidence interval for the actual number of days that it takes for the soda to dissolve a mouse and interpret the interval in the context of the problem. $n = 36, \bar{x} = 32, s = 5$. Since $n \geq 30$, we can use Z-distribution approximation ($Z_{0.005} = 2.576$). (Alternatively using T with $df = 35$, $t \approx 2.72$). Using Z:

$$32 \pm 2.576\left(\frac{5}{\sqrt{36}}\right) = 32 \pm 2.576(0.833) = 32 \pm 2.15$$

$$[29.85, 34.15] \text{ days}$$

We are 99% confident the true mean time to dissolve is between 29.85 and 34.15 days.

(b) If a person claims to have found a mouse in their can of Mountain Dew, 36 days after the soda was dispensed into the can, would they have a solid case against PepsiCola? Why or why not? If the mean time to dissolve is at most 34.15 days (upper end of 99% CI), then by 36 days, the mouse should have dissolved. Finding a whole mouse after 36 days is highly unlikely if Pepsi's data is correct. Thus, they likely do NOT have a solid case (it supports Pepsi's defense that the mouse would be gone).

(c) Construct and interpret a 95% upper confidence bound for the number of days that it takes for Mountain Dew to dissolve a mouse. $Z_{0.05} = 1.645$.

$$UB = 32 + 1.645(0.833) = 32 + 1.37 = \mathbf{33.37} \text{ days}$$

We are 95% confident that the mean time to dissolve is less than 33.37 days.

10. **Artificial Sweeteners**

The artificial sweetener sucralose makes you three times hungrier than sugar does.

A nutrition researcher questions this bold claim and decides to investigate. She recruits a sample of 20 participants who consume a fixed amount of sucralose and then report their hunger level on a standardized scale (0 to 100). The researcher finds that the sample mean hunger score is 71.4, with a sample standard deviation of 10.2.

Prior studies suggest that sugar typically produces an average hunger score of 68. Assume hunger scores are approximately normally distributed.

(a) At the 2% significance level, test whether sucralose has a different effect on hunger than sugar. Calculate the $P-$value, and write a conclusion in the context of the problem. $H_0 : \mu = 68$ vs $H_a : \mu \neq 68$. $n = 20, \bar{x} = 71.4, s = 10.2$. (Unknown sigma $\rightarrow$ T-test).

$$t = \frac{71.4 - 68}{10.2/\sqrt{20}} = \frac{3.4}{2.28} = 1.49$$

$df = 19$. Two-tailed test. From T-table: $t = 1.49$ is between $t_{0.10} = 1.328$ and $t_{0.05} = 1.729$. One-tail area is between 0.05 and 0.10. Two-tail P-value is between 0.10 and 0.20. (Approx **0.15**). Since $P > 0.02$, **fail to reject** H_0. There is no significant difference in hunger levels.

(b) Construct an appropriate interval estimate for the mean to corroborate the result from (a). Explain how this interval estimate supports the conclusion obtained from the hypothesis test.
Since $\alpha = 0.02$ (two-tailed), construct a 98% Confidence Interval. $t_{0.01,19} = 2.539$.

$$71.4 \pm 2.539(2.28) = 71.4 \pm 5.79$$

$$[65.61, 77.19]$$

Since 68 lies within this interval, we cannot reject the null hypothesis, supporting the conclusion in (a).