Diff Eq - Clex 10 - solutions

① $A = \begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix}$: $\lambda_1 = 0$, $\xi_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$; $\lambda_2 = -1$, $\xi_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$A = T \Lambda T^{-1}$ with $T = (\xi_1, \xi_2) = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$, $\Lambda = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$, $T^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$

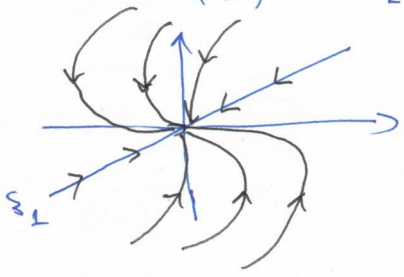
$\cos \pi A = T \begin{pmatrix} \cos \pi 0 & 0 \\ 0 & \cos \pi(-1) \end{pmatrix} T^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ 4 & -3 \end{pmatrix}$

② $\begin{vmatrix} 3-\lambda & -18 \\ 2 & -9-\lambda \end{vmatrix} = \lambda^2 + 6\lambda + 9 = (\lambda+3)^2$, $\lambda_1 = \lambda_2 = -3$

$\lambda = -3$ $\left(\begin{array}{cc|c} 6 & -18 & 0 \\ 2 & -6 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right)$ $\xi_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ 2nd eigenvector missing

$(A+3\mathbb{I})\eta = \xi_1$, $\left(\begin{array}{cc|c} 6 & -18 & 3 \\ 2 & -6 & 1 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & -3 & 1/2 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow \eta = \begin{pmatrix} 1/2 \\ 0 \end{pmatrix}$

$\vec{x}(t) = c_1 e^{-3t} \begin{pmatrix} 3 \\ 1 \end{pmatrix} + c_2 \left[t e^{-3t} \begin{pmatrix} 3 \\ 1 \end{pmatrix} + e^{-3t} \begin{pmatrix} 1/2 \\ 0 \end{pmatrix} \right]$



③ $A = \begin{pmatrix} 6 & 1 \\ 4 & 3 \end{pmatrix}$; $\lambda_1 = 2$, $\xi_1 = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$; $\lambda_2 = 7$, $\xi_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$T = \begin{pmatrix} 1 & 1 \\ -4 & 1 \end{pmatrix}$, $T^{-1} = \frac{1}{5} \begin{pmatrix} 1 & -1 \\ 4 & 1 \end{pmatrix}$. New variables $\vec{y} = T^{-1} \vec{x}$.

$\vec{y}' = \Lambda \vec{y} + T^{-1} \vec{g}$

$\vec{y}' = \begin{pmatrix} 2 & 0 \\ 0 & 7 \end{pmatrix} \vec{y} + \frac{1}{5} \begin{pmatrix} 1 & -1 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} 6t \\ -10t+4 \end{pmatrix}$

$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 2y_1 \\ 7y_2 \end{pmatrix} + \frac{1}{5} \begin{pmatrix} 16t - 4 \\ 14t + 4 \end{pmatrix}$

$y_1 = e^{2t} \int e^{-2t} \cdot \frac{1}{5} (16t - 4)\, dt = -\frac{8}{5} t - \frac{2}{5} + c_1 e^{2t}$

$y_2 = e^{7t} \int e^{-7t} \frac{1}{5} (14t + 4)\, dt = -\frac{2}{5} t - \frac{6}{35} + c_2 e^{7t}$

Diff eq - Clex 10 - solutions

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = T \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -4 & 1 \end{pmatrix} \begin{pmatrix} -8/5\, t - 2/5 + c_1 e^{2t} \\ -2/5\, t - 6/35 + c_2 e^{7t} \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = t \begin{pmatrix} -2 \\ 6 \end{pmatrix} + \frac{1}{7} \begin{pmatrix} -4 \\ 10 \end{pmatrix} + c_1 e^{2t} \begin{pmatrix} 1 \\ -4 \end{pmatrix} + c_2 e^{7t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\vec{x}(0) = \frac{1}{7} \begin{pmatrix} -4 \\ 10 \end{pmatrix} + c_1 \begin{pmatrix} 1 \\ -4 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \Rightarrow c_1 = \frac{7}{5}, \quad c_2 = \frac{76}{35}$$

$$\vec{x}(t) = t \begin{pmatrix} -2 \\ 6 \end{pmatrix} + \frac{1}{7} \begin{pmatrix} -4 \\ 10 \end{pmatrix} + \frac{7}{5} e^{2t} \begin{pmatrix} 1 \\ -4 \end{pmatrix} + \frac{76}{35} e^{7t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

(4)

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \quad \lambda_1 = -1, \ \xi_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}; \quad \lambda_2 = 3, \ \xi_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\Psi(t) = \begin{pmatrix} e^{-t} & e^{3t} \\ -e^{-t} & e^{3t} \end{pmatrix}, \quad \Psi^{-1}(t) = \frac{1}{2} \begin{pmatrix} e^{t} & -e^{t} \\ e^{-3t} & e^{-3t} \end{pmatrix}$$

$$\int \Psi^{-1}(t) \vec{g}(t)\, dt = \int \begin{pmatrix} 1/2\, e^{2t} + te^{t} \\ 1/2\, e^{-t} + te^{-3t} \end{pmatrix} dt =$$

$$= \begin{pmatrix} 1/6\, e^{3t} + te^{t} - e^{t} + c_1 \\ -1/2\, e^{-t} + 1/3\, te^{-3t} + 1/9\, e^{-3t} + c_2 \end{pmatrix}$$

$$\vec{x}(t) = \Psi(t) \int \Psi^{-1}(t) \vec{g}(t)\, dt =$$

$$= e^{2t} \begin{pmatrix} -1/3 \\ -2/3 \end{pmatrix} + t \begin{pmatrix} 4/3 \\ -2/3 \end{pmatrix} + \begin{pmatrix} -8/9 \\ 10/9 \end{pmatrix} + c_1 e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$