Diff Eq - Clex 13 - solutions

① $\mathcal{L}\{y''\} - 10\,\mathcal{L}\{y'\} + 9\,\mathcal{L}\{y\} = 5\,\mathcal{L}\{t\}$

$s^2Y(s) - s\,y(0) - y'(0) - 10s\,Y(s) + 10\,y(0) + 9Y(s) = \frac{5}{s^2}$

$(s^2 - 10s + 9)Y(s) + s - 12 = 5/s^2$

$Y(s) = \frac{5 + 12s^2 - s^3}{s^2(s-9)(s-1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-9} + \frac{D}{s-1}$

$5 + 12s^2 - s^3 = As(s-9)(s-1) + B(s-9)(s-1) + Cs^2(s-1) + Ds^2(s-9)$

$s=0 \Rightarrow B = 5/9$; $s=1 \Rightarrow D = -2$; $s=9 \Rightarrow C = \frac{31}{81}$; $s=2 \Rightarrow A = \frac{50}{81}$

$Y(s) = \frac{50}{81}\cdot\frac{1}{s} + \frac{5}{9}\,\frac{1}{s^2} + \frac{31}{81}\,\frac{1}{s-9} - 2\,\frac{1}{s-1}$

$y(t) = \frac{50}{81} + \frac{5}{9}t + \frac{31}{81}e^{9t} - 2e^{t}$

② $(s^2 - 6s + 15)Y(s) + s - 2 = \frac{6}{s^2+9}$

$Y(s) = \frac{-s^3 + 2s^2 - 9s + 24}{(s^2+9)(s^2-6s+15)} = \frac{As+B}{s^2+9} + \frac{Cs+D}{s^2-6s+15}$

$s^3: A + C = -1$, $s^2: -6A + B + D = 2$; $s^1: 15A - 6B + 9C = -9$

$s^0: 15B + 9D = 24 \Rightarrow A = \frac{1}{10}$, $B = \frac{1}{10}$, $C = -\frac{11}{10}$, $D = \frac{5}{2}$

$Y(s) = \frac{1}{10}\,\frac{s+1}{s^2+9} + \frac{1}{10}\,\frac{-11s+25}{s^2-6s+15}$

$\mathcal{L}^{-1}\left\{\frac{s+1}{s^2+9}\right\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2+9}\right\} = \cos 3t + \frac{1}{3}\sin 3t$

$\mathcal{L}^{-1}\left\{\frac{-11s+25}{s^2-6s+15}\right\} = \mathcal{L}^{-1}\left\{\frac{-11(s-3)-8}{(s-3)^2+6}\right\} = -11\,\mathcal{L}^{-1}\left\{\frac{(s-3)}{(s-3)^2+6}\right\} - 8\,\mathcal{L}^{-1}\left\{\frac{1}{(s-3)^2+6}\right\} =$

$= -11e^{3t}\cos\sqrt{6}t - \frac{8}{\sqrt{6}}\sin\sqrt{6}t$

$y(t) = \frac{1}{10}\cos 3t + \frac{1}{30}\sin 3t - \frac{11}{10}e^{3t}\cos\sqrt{6}t - \frac{4}{5\sqrt{6}}e^{3t}\sin\sqrt{6}t$

③ $f(t) = u_1(t)(t^2 - 2t + 2) = u_1(t)\left[(t-1)^2 + 1\right]$

$\mathcal{L}\{f(t)\} = \mathcal{L}\{u_1(t)(t-1)^2\} + \mathcal{L}\{u_1(t)\} = e^{-s}\frac{2}{s^3} + e^{-s}\frac{1}{s}$

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(4) $\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^2+s-2}\right\} = \mathcal{L}^{-1}\left\{e^{-2s}\left[\frac{1}{3}\frac{1}{s-1} - \frac{1}{3}\frac{1}{s+2}\right]\right\} =$

$= \frac{1}{3}\mathcal{L}^{-1}\left\{e^{-2s}\frac{1}{s-1}\right\} - \frac{1}{3}\mathcal{L}^{-1}\left\{e^{-2s}\frac{1}{s+2}\right\} = \frac{1}{3}u_2(t)\left[e^{t-2} - e^{-2(t-2)}\right]$

(5) $g(t) = u_\pi(t) - u_{2\pi}(t)$

$s^2 Y(s) - s - 4Y(s) = e^{-\pi s}\cdot\frac{1}{s} - e^{-2\pi s}\frac{1}{s}$

$Y(s) = \frac{e^{-\pi s} - e^{-2\pi s}}{s(s^2+4)} + \frac{s}{s^2+4}$

$\frac{1}{s(s^2+4)} = \frac{A}{s} + \frac{Bs+C}{s^2+4} \Rightarrow A = \frac{1}{4},\ B = -\frac{1}{4},\ C = 0$

$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{4}\frac{e^{-\pi s}}{s}\right\} + \mathcal{L}^{-1}\left\{-\frac{1}{4}\frac{e^{-\pi s}}{s^2+4}\right\} + \mathcal{L}^{-1}\left\{-\frac{1}{4}\frac{e^{-2\pi s}}{s}\right\} +$

$+ \mathcal{L}^{-1}\left\{\frac{1}{4}\frac{e^{-2\pi s}}{s^2+4}\right\} + \mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} =$

$= \frac{1}{4}u_\pi(t)\left\{1 - \cos 2(t-\pi)\right\} - \frac{1}{4}u_{2\pi}(t)\left\{1 - \cos 2(t-2\pi)\right\} + \cos 2t$

$y(t) = \frac{1}{4}\left\{u_\pi(t) - u_{2\pi}(t)\right\}(1 - \cos 2t) + \cos 2t.$