Diff Eq - Clex 12 - Solutions

(1) Let $J(s) = \mathcal{L}\{I(t)\}$. Applying Laplace transform to both sides of the DE we obtain:

$$s^2 J(s) + 4J(s) = \frac{1}{s} - \frac{2e^{-s}}{s} + \frac{e^{-2s}}{s}$$

Here we used the fact that the $g(t) = 1 - 2u_1(t) + u_2(t)$.

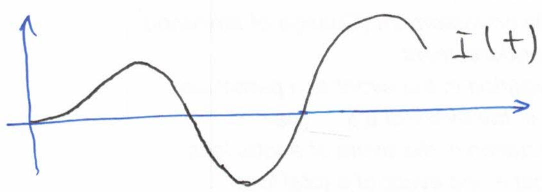
$$J(s) = \frac{1}{s(s^2+4)} \left[1 - 2e^{-s} + e^{-2s}\right]$$

$$\frac{1}{s(s^2+4)} = \frac{1}{4}\cdot\frac{1}{s} - \frac{1}{4}\frac{s}{s^2+4}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+4)}\right\} = \frac{1}{4} - \frac{1}{4}\cos 2t$$

$$I(t) = \mathcal{L}^{-1}\left\{\left(\frac{1}{s(s^2+4)}\right)\cdot\left[1-2e^{-s}+e^{-2s}\right]\right\} =$$

$$= \left(\frac{1}{4} - \frac{1}{4}\cos 2t\right) - \left(\frac{1}{2} - \frac{1}{2}\cos 2(t-1)\right)u_1(t) + \left(\frac{1}{4} - \frac{1}{4}\cos 2(t-2)\right)u_2(t)$$



(2)

$$sX(s) - 4 - 2Y(s) = \frac{4}{s^2}$$

$$sY(s) + 5 + 2Y(s) - 4X(s) = -\frac{4}{s^2} - \frac{2}{s}$$

$$\Rightarrow \quad sX(s) - 2Y(s) = \frac{4s^2+4}{s^2}, \quad -4X(s) + (s+2)Y(s) = -\frac{5s^2+2s+4}{s^2}$$

Eliminate $Y(s)$ from the system:

$$[s(s+2) - 8]\,X(s) = (s+2)\frac{4s^2+4}{s^2} - \frac{10s^2-4s-8}{s^2}$$

$$X(s) = \frac{4s-2}{(s+4)(s-2)} = \frac{3}{s+4} + \frac{1}{s-2}$$

$$x(t) = 3e^{-4t} + e^{2t}$$

Solving for $y(t)$ in the first DE of the system we have:

$$y(t) = \frac{1}{2}x'(t) - 2t = -6e^{-4t} + e^{2t} - 2t$$

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(3) $$\int_{-\infty}^{\infty} f(t)\left[\frac{d}{dt}\delta(t)\right]dt = \cancel{f(t)\delta(t)\Big|_{-\infty}^{\infty}}_{0} - \int_{-\infty}^{\infty}\left[\frac{d}{dt}f\right](t)\,\delta(t)\,dt =$$

$$= \int_{-\infty}^{\infty} f'(t)\,\delta(t)\,dt = f'(0)$$

By induction. The case $n=1$ was proven above. Assume the identity holds for n. We will prove it for $n+1$.

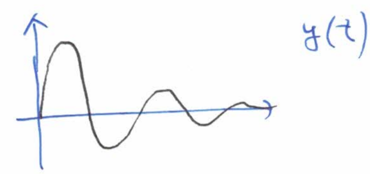
$$\int_{-\infty}^{\infty} f(t)\,\delta^{(n+1)}(t)\,dt = \int_{-\infty}^{\infty} f(t)\left[\frac{d}{dt}\delta^{(n)}(t)\right]dt =$$

$$= \cancel{f(t)\,\delta^{(n)}(t)\Big|_{-\infty}^{\infty}}_{0} - \int_{-\infty}^{\infty} f'(t)\,\delta^{(n)}(t)\,dt = -(-1)^n f'^{(n)}(0) =$$

$$= (-1)^{n+1} f^{(n+1)}(0)$$

(4) $$\mathcal{L}\{2y'' + y' + 2y\} = \mathcal{L}\{\delta(t)\} \Rightarrow (2s^2+s+2)Y(s) = 1$$

$$Y(s) = \frac{1}{2s^2+s+2} = \frac{1}{2}\,\frac{1}{(s+1/4)^2 + \frac{15}{16}}$$



$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \frac{2}{\sqrt{15}}\,e^{-t/4}\sin\frac{\sqrt{15}}{4}t$$

(5) $$Lq'' + Rq' + \frac{1}{C}q = E(t)$$

$$q'' + 8q' + 15q = 4\delta(t-2), \quad q(0)=1, \; q'(0)=0$$

$\mathcal{L} \to$ $$s^2Q(s) - s + 8sQ - 8 + 15Q = 4e^{-2s}$$

$$Q(s) = \frac{s+8}{s^2+8s+15} + e^{-2s}\frac{4}{s^2+8s+15} = \frac{s+8}{(s+3)(s+5)} + e^{-2s}\frac{4}{(s+3)(s+5)}$$

$$Q(s) = \frac{5}{2}\,\frac{1}{s+3} - \frac{3}{2}\,\frac{1}{s+5} + e^{-2s}\left[\frac{2}{s+3} - \frac{2}{s+5}\right]$$

$$q(t) = \frac{5}{2}e^{-3t} - \frac{3}{2}e^{-5t} + u_2(t)\left[2e^{-3(t-2)} - 2e^{-5(t-2)}\right]$$